WARMUP!! Find the general solution to the logistic differential equation below. Your answer should be in the form P = f(t). Keep in mind that k and L are constants. (Hint: you might need to use partial fractions.)

$$\frac{dP}{dt} = kP(L-P)$$

$$\int \left(\frac{1}{LP} + \frac{1}{L(L-P)}\right) dP = \int k\, dt$$

$$\int \left(\frac{1}{P} + \frac{1}{L-P}\right) dP = \int Lk\, dt$$

$$\ln|P| - \ln|L-P| = Lkt + C$$

$$\ln\frac{|P|}{|L-P|} = Lkt + C$$

$$\left|\frac{P}{L-P}\right| = e^{Lkt+C} = Ae^{Lkt}$$

$$\frac{P}{L-P} = Me^{Lkt}$$

$$P = MLe^{Lkt} - MPe^{Lkt}$$

$$P(1 + Me^{Lkt}) = MLe^{Lkt}$$

$$P = \frac{MLe^{Lkt}}{1 + Me^{Lkt}} = \frac{L}{be^{-Lkt} + 1}$$

$$P = \frac{L}{1 + be^{-Lkt}}$$

6.3 Logistic Equations!

Essential Learning Target:

Knows that the model for logistic growth that arises from the statement, "The rate of change of a quantity is jointly proportional to the size of the quantity and the difference between the quantity and the carrying capacity" is

$$\frac{dy}{dt} = ky(a - y)$$

I know what you're thinking about the warmup, "Wow! That was fun! I totally see how I can apply this to my own life!"

Actually, what you derived was called a logistic growth model.

To generate a traditional growth model describing population, we started with the knowledge that the rate of growth of the population was proportional to the population, dy/dt = ky, and then solved to get $y = Ce^{kt}$.

However, true populations are often limited and cannot grow without bound forever as the $y = Ce^{kt}$ function describes. The upper limit is called L and is known as the carrying capacity.

The new growth model that incorporates this limiting condition is called a logistic differential equation and is written as:

$$\frac{dP}{dt} = kP(L - P)$$

Because of this, anytime you see a logistic differential equation in the form:

College Board's model

$$\frac{dy}{dt} = ky(L - y)$$

k = $\frac{\text{growth factor}}{\text{carrying capacity}}$

L = carrying capacity

or

Book's model

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$$

k = growth constant

L = carrying capacity

You can assume that the solution will be in the form:

$$y = \frac{L}{1 + be^{-Lkt}}$$

or

$$y = \frac{L}{1 + be^{-kt}}$$

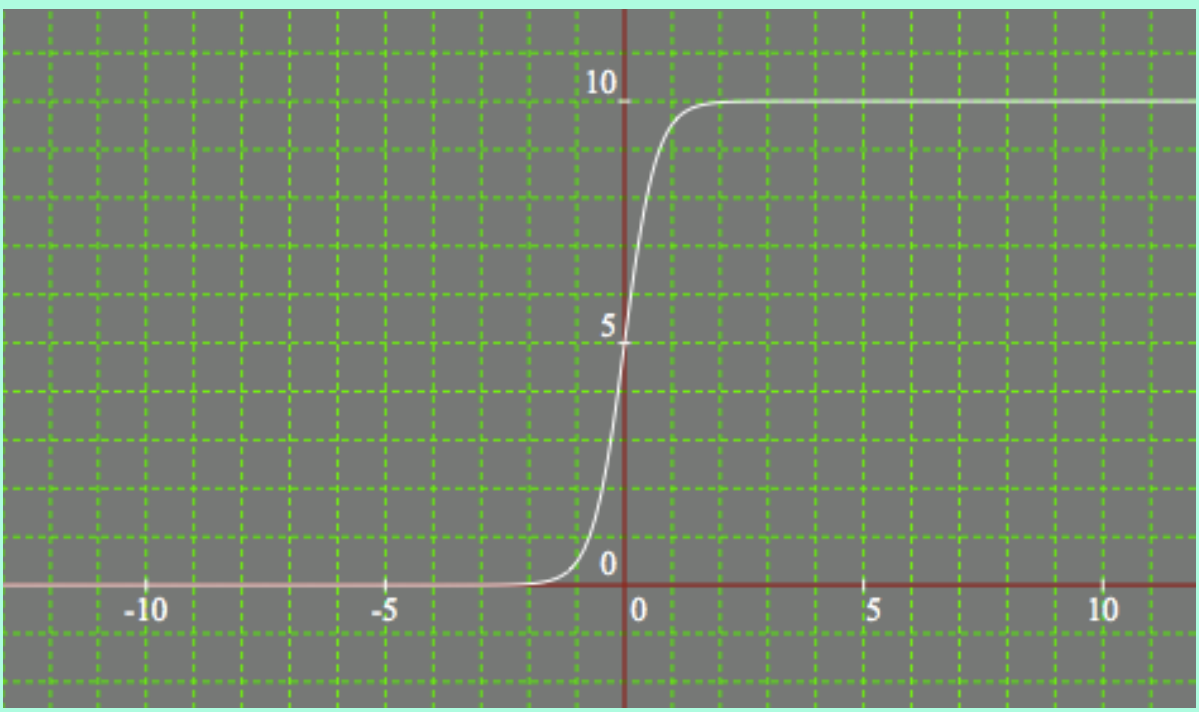
To graph a logistic equation:

1) minimum value is 0

2) maximum value is L

3) point of inflection is at $\left(\frac{\ln b}{LK}, \frac{1}{2}L\right)$

ex) $y = \dfrac{10}{1 + e^{-3x}}$ is graphed below



Let S represent sales of a new product (in thousands of units per month), let L represent the maximum level of sales (in thousands of units), and let t represent time (in months). Given that the rate of change of S with respect to t varies jointly as a product of S and (L - S),

a) Write a differential equation for the sales model in terms of S, k and L

✓ $$\frac{dS}{dt} = kS(L - S)$$

b) Write the solution to the differential equation, expressing S as a function of t, given that L = 100, S(0) = 10 and S(1) = 20.

✓ $$S = \frac{L}{1 + Ae^{-Lkt}} = \frac{100}{1 + Ae^{-100kt}}$$

$$S(0) = \frac{100}{1 + A} = 10 \quad so\ A = 9$$

$$\text{So now } S(t) = \frac{100}{1 + 9e^{-100kt}}$$

$$S(1) = \frac{100}{1 + 9e^{-100k}} = 20 \quad so\ \ 5 = 1 + 9e^{-100k}$$

$$k = -\frac{1}{100}\ln\left(\frac{4}{9}\right)$$

$$So\ now\ S(t) = \frac{100}{1 + 9e^{-100\cdot -\frac{1}{100}\ln\left(\frac{4}{9}\right)t}} = \frac{100}{1 + 9e^{\left(\ln\frac{4}{9}\right)t}}$$

c) At what time is the growth in sales increasing most rapidly?

Logistic functions are growing most rapidly at their point of inflection.

✓ $$t = \frac{\ln b}{LK} = \frac{\ln 9}{-\ln\frac{4}{9}} \approx 2.7095 \text{ months}$$

If you can't remember the poi formula, just remember it occurs at the 'halfway' y-value, so set S = 50 and solve for t ex) If the rate of change of a population is modeled by the function below, state the values of k and L. Then write a model for the population (without solving for b).

$$\frac{dP}{dt} = 5P\left(1 - \frac{P}{1000}\right)$$

To get this in the correct format, factor out a 1/1000:

✓

$$so\ \frac{dP}{dt} = \frac{1}{200}P(1000 - P)$$

$$\text{now } k = \frac{1}{200} \text{ and } L = 1000$$

$$so\ P = \frac{1000}{1 + be^{-5t}}$$

ex) If a population is modeled by the equation below, state the values of k, b, L, and the initial population.

$$P(t) = \frac{6666}{1 + 10e^{-0.3t}}$$

$$P = \frac{L}{1 + be^{-Lkt}}$$

✓

b = 10, L = 6666

6666k = 0.3, so k ≈ 0.000045

$$P(0) = \frac{6666}{1 + 10} = 606$$

ex) A state game commission releases 40 elk into a game refuge. After 5 years, the elk population is 104. The commission believes that the environment can support no more than 4000 elk.

a) Write a logistic differential equation for the growth of the elk population

$$\frac{dP}{dt} = kP(4000 - P)$$

b) Write a model for the elk population in terms of t

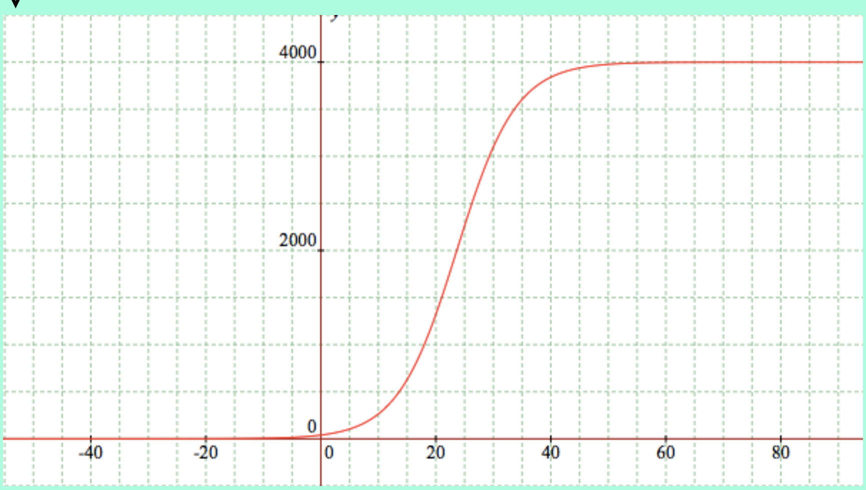
$$P = \frac{4000}{1+be^{-4000kt}}$$

$$P(0) = \frac{4000}{1+b} = 40 \text{ so } b = 99$$

$$P(5) = \frac{4000}{1+99e^{-20000k}} = 104 \text{ so } k = -\frac{1}{20000}\ln\left(\frac{487}{1287}\right)$$

$$P(t) = \frac{4000}{1+99e^{\frac{1}{5}\ln\left(\frac{487}{1287}\right)t}}$$

c) Sketch a graph of the population model



d) Find the limit of the population model as t approaches ∞

$$\lim_{t\to\infty} \frac{4000}{1+99e^{\frac{1}{5}\ln\left(\frac{487}{1287}\right)t}} = \frac{4000}{1+0} = 4000$$

$$or \lim_{t\to\infty} y = 4000$$

by the Kari Satterness method

1991 BC exam #6 (no calculator)

A certain rumor spreads through a community at the rate of dy/dt = 2y(1 - y), where y is the proportion of the population that has heard the rumor at time t.

a) What proportion of the community has heard the rumor when it is spreading the fastest?

a) Since the rate of rumor spreading, $\frac{dy}{dt} = 2y - y^2$, it will be spreading the fastest at its max, which is when its derivative = 0.

so $\frac{d^2y}{dy^2} = 2\frac{dy}{dt} - 4y\frac{dy}{dt} = 2(2y - 2y^2) - 4y(2y - 2y^2)$

$= 4y - 4y^2 - 8y^2 + 8y^3 = 4y - 12y^2 + 8y^3 = 4y(2y - 1)(y - 1)$

so this = 0 at y = 0, ½, or 1. Doing a quick sign line yields that the max occurs at y = ½. Since y is the proportion that has heard the rumor, the percentage is ½ or 50%.

✓

b) If at time t = 0, ten percent of the people have heard the rumor, find y as a function of t.

b) $\frac{dy}{dt} = 2y(1 - y)$

Since this fits our logistic differential equation model with k = 2 and L = 1, the general solution would be: $y = \frac{1}{1+be^{-2t}}$

plugging in (0, 1/10) we get: $\frac{1}{10} = \frac{1}{1+b}$

so b = 9 and our final logistic growth model is: $y = \frac{1}{1+9e^{-2t}}$

✓

c) At what time t is the rumor spreading the fastest?

c) Since the rumor is spreading the fastest when y = ½, plug this in and solve for t to get:

$\frac{1}{2} = \frac{1}{1+9e^{-2t}}$ so $2 = 1 + 9e^{-2t}$

$e^{-2t} = \frac{1}{9}$ so $t = -\frac{1}{2}\ln\frac{1}{9} = -\frac{1}{2}(\ln 1 - \ln 9) = \frac{1}{2}\ln 9 = \ln 3$

✓

Given the differential equation $\frac{dy}{dt} = y\left(6 - \frac{y}{50}\right)$

where y(0) = 25,

What is the $\lim_{t \to \infty} y(t)$?

$$\frac{dy}{dt} = \frac{1}{50} y (300 - y)$$

So $\lim_{t \to \infty} y(t) = \underline{300}$

Find the logistic equation that satisfies the initial condition $\frac{dy}{dt} = \frac{3y}{20} - \frac{y^2}{1600}$ and passes through the point (0, 15)

$$\frac{dy}{dt} = \frac{1}{1600} y \left(\frac{3(1600)}{20} - y \right)$$

$$= \frac{1}{1600} y (240 - y)$$

$$y = \frac{240}{1 + be^{-\frac{1}{1600} \cdot 240 \cdot t}} = \frac{240}{1 + be^{-\frac{3}{20} t}}$$

$$y(0) = \frac{240}{1+b} = 15$$

so $b = 15$

$$y = \frac{240}{1 + 15e^{-\frac{3}{20} t}}$$

It's Warmup time!

2013 BC free response #1

(calculators permitted)

On a certain workday, the rate, in tons per hour, at which unprocessed gravel arrives at a gravel processing plant is modeled by $G(t) = 90 + 45\cos(t^2/18)$, where t is measured in hours and $0 \leq t \leq 8$. At the beginning of the workday ($t = 0$), the plant has 500 tons of unprocessed gravel. During the hours of operation, $0 \leq t \leq 8$, the plant processes gravel at a constant rate of 100 tons per hour.

a) Find $G'(5)$. Using correct units, interpret your answer in the context of the problem.

b) Find the total amount of unprocessed gravel that arrives at the plant during the hours of operation on this workday.

c) Is the amount of unprocessed gravel at the plant increasing or decreasing at time $t = 5$ hours? Show the work that leads to your answer.

d) What is the maximum amount of unprocessed gravel at the plant during the hours of operation on this workday? Justify your answer.

What have we learned?

Can I:

- recognize a logistic differential equation?
- sketch a rough graph of a logistic equation?
- find a logistic model based on a logistic differential equation and some given info?
- find the limit as a logistic equation approaches infinity?

$$\frac{1}{(x^2+4)^2} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2}$$

February 03, 2017