Answer Key

A.P. Calculus AB **Review of Derivatives**

Use the general definition of the derivative to find $\frac{dy}{dx}$ if:

1. $f(x) = 5x^2 - 3$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{5(x+h)^2 - 3 - (5x^2 - 3)}{h}$$
$$= \lim_{h \to 0} \frac{5(x^2 + 2xh + h^2) - 3 - 5x^2 + 3}{h}$$
$$= \lim_{h \to 0} \frac{5x^2 + 10xh + 5h^2 - 3 - 5x^2 + 3}{h} = \lim_{h \to 0} \frac{10x\cancel{h} + 5h^2}{\cancel{h}} = \boxed{10x}$$

2. $f(x) = \frac{1}{x}$

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$
$$= \lim_{h \to 0} \frac{\frac{x - (x+h)}{x(x+h)}}{h}$$
$$= \lim_{h \to 0} \frac{-\cancel{h}}{x\cancel{h}(x+h)} = \boxed{\frac{-1}{x^2}}$$

Use the definition of the derivative at a specific point to find $g'(c)$ if:

3. $g(x) = \sqrt{20 - x}$, c = 4

$$g'(4) = \lim_{x \to 4} \frac{\sqrt{20-x} - 4}{x - 4} \cdot \frac{\sqrt{20-x} + 4}{\sqrt{20-x} + 4}$$
$$= \lim_{x \to 4} \frac{20 - x - 16}{(x-4)(\sqrt{20-x} + 4)} = \lim_{x \to 4} \frac{-(x-4)}{(x-4)(\sqrt{20-x} + 4)}$$
$$= \boxed{\frac{-1}{8}}$$

4. $g(x) = 5x^2 + x$, c = -1

$$g'(-1) = \lim_{x \to -1} \frac{5x^2 + x - 4}{x + 1}$$
$$= \lim_{x \to -1} \frac{(5x-4)\cancel{(x+1)}}{\cancel{x+1}}$$
$$= \boxed{-9}$$

Write an equation of the tangent line to the graph of $y = f(x)$ at $x = c$ if:

5. $f(x) = x^2 + 3x$, c = -2

$f(-2) = -2$

$f'(x) = 2x + 3$

$f'(-2) = -1$

$\boxed{y + 2 = -1(x + 2)}$

6. $f(x) = 5x - x^2$, $c = 1$

$f(1) = 4$

$f'(x) = 5 - 2x$

$f'(1) = 3$

$\boxed{y - 4 = 3(x - 1)}$

7. A mass is bouncing up and down on a spring hanging from the ceiling. Its distance, y, in feet from the ceiling is measured by strobe photography each $\frac{1}{10}$ of a second, giving the values in the table below, where t is the time in seconds.

t	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00	1.10
y	3.99	5.84	7.37	8.00	7.48	6.01	4.16	2.63	2.00	2.52

a. Tell what the average velocity of the mass is from t = 0.2 seconds to t = 0.4 seconds.

$$\frac{7.37 - 3.99}{.4 - .2} = 16.9 \frac{ft}{sec}$$

b. Does the mass appear to be going up or going down during this timeframe?

up b/c $v(t) > 0$

3. How fast is the distance changing at t = 0.3 seconds? t = 0.6 seconds? t = 1.0 seconds?

$t = .3$: $\frac{7.37 - 3.99}{.4 - .2} = \boxed{16.9 \frac{ft}{sec}}$ $t = .6$: $\frac{6.01 - 8}{.7 - .5} = \boxed{-9.95 \frac{ft}{sec}}$ $t = 1$: $\frac{2.52 - 2.63}{1.1 - .9} = \boxed{-.55 \frac{ft}{sec}}$

4. For each of the times in problem 3, is the mass going up or down?

$t = .3 \to$ up, $t = .6 \to$ down, $t = 1 \to$ down

10. Liquid is being poured into a large vat. After t hours, the amount of gallons of liquid in the vat can be represented by $V(t) = 5t - \sqrt{t}$.

a) What is the average rate of liquid poured into the vat over the first 4 hours? (Include units of measure)

$$\frac{V(4)-V(0)}{4-0} = \frac{20-2-0}{4} = \frac{18}{4} = \frac{9}{2}\ \frac{\text{gallons}}{\text{hour}}$$

b) At what rate is the liquid being poured into the vat when $t = 4$? (Include units of measure)

$$V'(t) = 5 - \tfrac{1}{2}t^{-\frac{1}{2}} = 5 - \frac{1}{2\sqrt{t}}$$

$$V'(4) = 5 - \frac{1}{4} = \frac{19}{4}\ \frac{\text{gallons}}{\text{hour}}$$

Use the product and quotient rules to differentiate the functions below. Express your answers in simplest factored form.

11. $f(x) = (x+3)(2x-1)$

$$f'(x) = 1(2x-1) + (x+3)(2)$$
$$= 2x - 1 + 2x + 6$$
$$= \boxed{4x+5}$$

12. $f(x) = (4x^3-3)(3x^2-1)$

$$f'(x) = (12x^2)(3x^2-1) + (4x^3-3)(6x)$$
$$= 36x^4 - 12x^2 + 24x^4 - 18x$$
$$= 60x^4 - 12x^2 - 18x$$
$$= \boxed{6x(10x^3-2x-3)}$$

13. $g(x) = 5x\sqrt{x}$

$$g'(x) = 5x^{\frac{1}{2}} + 5x\left(\tfrac{1}{2}x^{-\frac{1}{2}}\right)$$
$$= 5\sqrt{x} + \frac{5x}{2\sqrt{x}}$$
$$= \frac{10x+5x}{2\sqrt{x}} = \boxed{\frac{15x}{2\sqrt{x}}}$$

14. $f(x) = \dfrac{3x}{x+5}$

$$f'(x) = \frac{3(x+5) - 3x(1)}{(x+5)^2}$$
$$= \frac{3x+15-3x}{(x+5)^2} = \boxed{\frac{15}{(x+5)^2}}$$

15. $g(y) = \dfrac{4y-3}{3-2y}$

$$g'(y) = \frac{4(3-2y) - (4y-3)(-2)}{(3-2y)^2}$$
$$= \frac{12 - 8y + 8y - 6}{(3-2y)^2} = \boxed{\frac{6}{(3-2y)^2}}$$

16. $f(x) = \dfrac{2x\sqrt{x}}{3x+5} = \dfrac{2x^{\frac{3}{2}}}{3x+5}$

$$f'(x) = \frac{3x^{\frac{1}{2}}(3x+5) - 2x^{\frac{3}{2}}(3)}{(3x+5)^2}$$
$$= \frac{9x^{\frac{3}{2}} + 15x^{\frac{1}{2}} - 6x^{\frac{3}{2}}}{(3x+5)^2}$$
$$= \frac{3x^{\frac{3}{2}} + 15x^{\frac{1}{2}}}{(3x+5)^2}$$
$$= \boxed{\frac{3\sqrt{x}(x+5)}{(3x+5)^2}}$$

17. Find the equation of the tangent line to $y = \dfrac{2-x}{5+x}$ at $x = 1$.

point: $(1, \frac{1}{6})$

$$y' = \frac{-1(5+x) - (2-x)(1)}{(5+x)^2} \qquad y'(1) = \frac{-6-1}{6^2} = \frac{-7}{36} \qquad \boxed{y - \tfrac{1}{6} = \tfrac{-7}{36}(x-1)}$$

Find the derivatives of each of the following. Express your answers in simplest factored form.

18. $f(x) = (4x^2+3)^3(3x-2)^2$

$$f'(x) = 3(4x^2+3)^2(8x)(3x-2)^2 + (4x^2+3)^3(2)(3x-2)(3)$$
$$= 24x(4x^2+3)^2(3x-2)^2 + 6(4x^2+3)^3(3x-2)$$
$$= 6(4x^2+3)^2(3x-2)(36x^2-8x+18)$$
$$= \boxed{12(4x^2+3)^2(3x-2)(18x^2-4x+9)}$$

19. $h(u) = \sqrt{u-1} \cdot \sqrt[3]{2u+3}$

$$h'(u) = \tfrac{1}{2}(u-1)^{-\frac{1}{2}}(2u+3)^{\frac{1}{3}} + (u-1)^{\frac{1}{2}}\left(\tfrac{1}{3}\right)(2u+3)^{-\frac{2}{3}}(2)$$
$$= \frac{\sqrt[3]{2u+3}}{2\sqrt{u-1}} + \frac{2\sqrt{u-1}}{3\sqrt[3]{(2u+3)^2}}$$
$$= \frac{3(2u+3) + 4(u-1)}{6\sqrt{u-1}\,\sqrt[3]{(2u+3)^2}} = \boxed{\frac{5(2u+1)}{6\sqrt{u-1}\,\sqrt[3]{(2u+3)^2}}}$$

20. $p(x) = \dfrac{x^2+10x+25}{x^2-10x+25}$

$$p'(x) = \frac{(2x+10)(x^2-10x+25) - (x^2+10x+25)(2x-10)}{(x^2-10x+25)^2}$$
$$= \frac{2x^3-10x^2-50x+250 - 2x^3-10x^2+50x+250}{(x^2-10x+25)^2}$$
$$= \frac{-20x^2+500}{(x-5)^4}$$
$$= \frac{-20(x+5)(x-5)}{(x-5)^4} = \boxed{\frac{-20(x+5)}{(x-5)^3}}$$

21. Find the equation of the tangent line to the curve $y = \sqrt{x^2+3}$ at the point (2, 1).

$$y' = \tfrac{1}{2}(x^2+3)^{-\frac{1}{2}}(2x)$$
$$= \frac{x}{\sqrt{x^2+3}} \qquad y'(2) = \frac{2}{\sqrt{7}}$$
$$\boxed{y - 1 = \frac{2}{\sqrt{7}}(x-2)}$$

22. Find the equation of the tangent line to the curve $y = \sqrt{3x-1}$ that is perpendicular to the line $3y + 2x = 3$.

$3y = -2x+3$ $\perp$ slope $= \frac{3}{2}$ $y' = \frac{1}{2}(3x-1)^{-\frac{1}{2}}(3) = \frac{3}{2\sqrt{3x-1}} = \frac{3}{2}$

$y = -\frac{2}{3}x + 1$

$\sqrt{3x-1} = 1$ point: $(\frac{2}{3}, 1)$

$3x-1 = 1$

$3x = 2$ $\boxed{y - 1 = \frac{3}{2}(x - \frac{2}{3})}$

$x = \frac{2}{3}$

Use implicit differentiation to find $\frac{dy}{dx}$.

23. $3x^2 - y^2 = 1$

$6x - 2y\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} = 6x$

$\boxed{\frac{dy}{dx} = \frac{3x}{y}}$

24. $x^5 + 3xy - y^5 = -3$

$5x^4 + 3y + 3x\frac{dy}{dx} - 5y^4\frac{dy}{dx} = 0$

$3x\frac{dy}{dx} - 5y^4\frac{dy}{dx} = -5x^4 - 3y$

$\frac{dy}{dx}(3x - 5y) = -5x^4 - 3y$

$\boxed{\frac{dy}{dx} = \frac{-5x^4 - 3y}{3x - 5y}}$

25. $x^3 = \frac{x-3y}{x+3y} \rightarrow x^4 + 3x^3y = x - 3y$

$4x^3 + 9x^2y + 3x^3\frac{dy}{dx} = 1 - 3\frac{dy}{dx}$

$3x^3\frac{dy}{dx} + 3\frac{dy}{dx} = 1 - 4x^3 - 9x^2y$

$\boxed{\frac{dy}{dx} = \frac{1 - 4x^3 - 9x^2y}{3x^3 + 3}}$

26. $\sqrt{y} = 2x^2 - 3y$

$\frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = 4x - 3\frac{dy}{dx}$

$\frac{1}{2\sqrt{y}}\frac{dy}{dx} + 3\frac{dy}{dx} = 4x$

$\frac{dy}{dx}\left(\frac{1}{2y} + 3\right) = 4x$

$\frac{dy}{dx} = \frac{4x}{\frac{1}{2y} + 3} = \boxed{\frac{8xy}{1 + 6y}}$

27. Find an equation of the tangent line to the circle $x^2 + y^2 = 9$ at the point where $x = 1$ in quadrant I.

point: $1 + y^2 = 9$ slope: $2x + 2y\frac{dy}{dx} = 0$

$y^2 = 8$ $\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$ $\boxed{y - \sqrt{8} = \frac{-1}{\sqrt{8}}(x - 1)}$

$y = \sqrt{8}$

$\frac{dy}{dx}$ at $(1, \sqrt{8}) = \frac{-1}{\sqrt{8}}$

28. Find the rate of change of y with respect to x at the point where $y = 1$ if $2y^2 - xy^5 = 3$.

$4y\frac{dy}{dx} - y^5 - x(5y^4)\frac{dy}{dx} = 0$ if $y = 1$, $2 - x = 3$

$x = -1$

$4(1)\frac{dy}{dx} - 1 - (-1)(5)\frac{dy}{dx} = 0$

$4\frac{dy}{dx} - 1 + 5\frac{dy}{dx} = 0 \longrightarrow 9\frac{dy}{dx} = 1 \longrightarrow \boxed{\frac{dy}{dx} = \frac{1}{9}}$

29. At what point(s) on the circle $x^2 - 4x + y^2 + 6y = 12$ is the tangent line parallel to the x-axis?

$2x - 4 + 2y\frac{dy}{dx} + 6\frac{dy}{dx} = 0$ $4 - 8 + y^2 + 6y = 12$

$\frac{dy}{dx}(2y + 6) = 4 - 2x$ $4 - 2x = 0$ $y^2 + 6y - 16 = 0$ $\boxed{(2,-8)\ (2,2)}$

$\frac{dy}{dx} = \frac{4 - 2x}{2y + 6} = 0 \longrightarrow$ $x = 2$ $(y + 8)(y - 2) = 0$

$y = -8, 2$