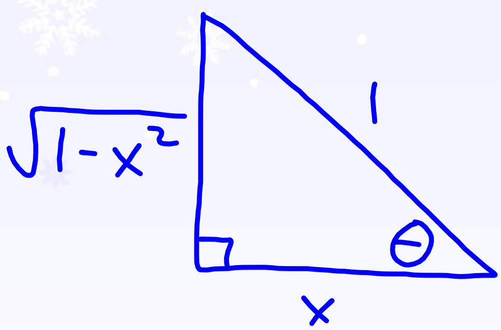
Let's warm up with a warmup!!

Draw a triangle and label the sides so that one of the sides ends up being $\sqrt{1 - x^2}$



$\int \sqrt{1-x^2}\,dx$?

8.4 Trig Substitution!!

At the end of this lesson you will be able to:

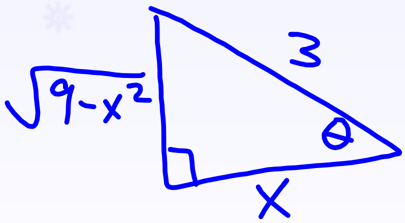
- Evaluate integrals involving roots of quadratics using trig identities

Let's walk through an example together:

$$\int \frac{dx}{x^2\sqrt{9-x^2}} =$$

1. Draw a triangle - then look at the integral and label the sides based on how it seems to fit

$a^2 - b^2$ => let a be the hypotenuse and b be one of the sides



$\sin\theta = \frac{\sqrt{9-x^2}}{3}$ $\qquad$ $\cos\theta = \frac{x}{3}$

$\sqrt{9-x^2} = 3\sin\theta$ $\qquad$ $x = 3\cos\theta$

$x^2 = 9\cos^2\theta$

2. Rewrite all of the parts of the integral in terms of the trig functions, then rewrite the integral

$dx = -3\sin\theta\, d\theta$

$$\int \frac{dx}{x^2\sqrt{9-x^2}} = \int \frac{-\cancel{3\sin\theta}\, d\theta}{(\cancel{3\sin\theta})(9\cos^2\theta)}$$

3. Evaluate the integral - you may have to use some trig methods you have learned in previous units

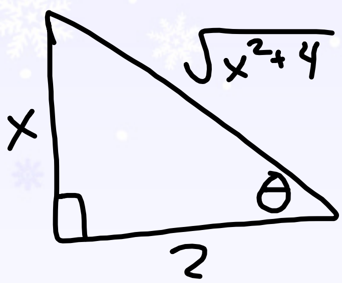
$$\int -\frac{1}{9}\cdot\frac{1}{\cos^2\theta}\, d\theta = \int -\frac{1}{9}\sec^2\theta\, d\theta = -\frac{1}{9}\tan\theta + C$$

4. Make sure you convert the answer back to algebraic form

$$= -\frac{1}{9}\left(\frac{\sqrt{9-x^2}}{x}\right) + C$$

$$\int \frac{1}{\sqrt{3x^2+12}}\,dx =$$

$$\frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{x^2+4}}\,dx$$



$$\tan\theta = \frac{x}{2} \qquad \sec\theta = \frac{\sqrt{x^2+4}}{2}$$

$$x = 2\tan\theta \qquad \sqrt{x^2+4} = 2\sec\theta$$

$$dx = 2\sec^2\theta\,d\theta$$

$$\frac{1}{\sqrt{3}}\int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta = \frac{1}{\sqrt{3}}\int \sec\theta\,d\theta$$

$$= \frac{1}{\sqrt{3}}\ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C$$

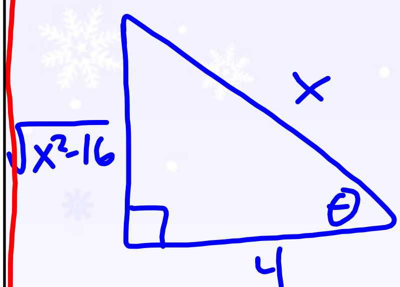
$$= \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{x^2+4}+x}{2}\right| + C$$

$$= \frac{1}{\sqrt{3}}\left(\ln\left|\sqrt{x^2+4}+x\right| - \ln 2\right) + C$$

$$= \frac{1}{\sqrt{3}}\ln\left|\sqrt{x^2+4}+x\right| + C$$

You try again!

$$\int \frac{1}{\sqrt{x^2-16}}\,dx =$$



$$\tan\theta = \frac{\sqrt{x^2-16}}{4} \qquad \sec\theta = \frac{x}{4}$$

$$\sqrt{x^2-16} = 4\tan\theta \qquad x = 4\sec\theta$$

$$dx = 4\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\cancel{4}\sec\theta\,\cancel{\tan\theta}\,d\theta}{\cancel{4}\,\cancel{\tan\theta}} = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$

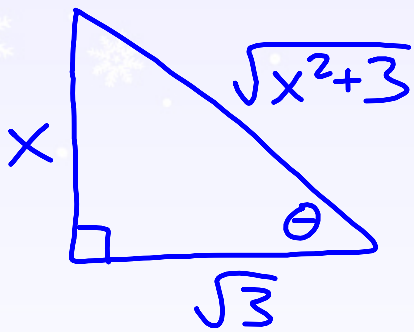
$$= \ln\left|\frac{x}{4} + \frac{\sqrt{x^2-16}}{4}\right| + C$$

$$= \ln\left|x + \sqrt{x^2-16}\right| - \ln 4 + C$$

$$= \ln\left|x + \sqrt{x^2-16}\right| + C$$

One more!

$$\int \frac{1}{(x^2+3)^{\frac{3}{2}}}\,dx = \int \frac{1}{\left(\sqrt{x^2+3}\right)^3}\,dx$$



$\tan\theta = \frac{x}{\sqrt{3}}$ $\qquad$ $\sec\theta = \frac{\sqrt{x^2+3}}{\sqrt{3}}$

$x = \sqrt{3}\tan\theta$ $\qquad$ $\sqrt{x^2+3} = \sqrt{3}\sec\theta$

$dx = \sqrt{3}\sec^2\theta\,d\theta$

$$= \int \frac{\sqrt{3}\sec^2\theta\,d\theta}{\left(\sqrt{3}\sec\theta\right)^3} = \int \frac{\sqrt{3}\sec^2\theta\,d\theta}{3\sqrt{3}\sec^3\theta}$$

$$= \frac{1}{3}\int \cos\theta\,d\theta = \frac{1}{3}\sin\theta + C$$

$$= \frac{1}{3}\frac{x}{\sqrt{x^2+3}} + C$$

$$\int \frac{1}{x^2-9}\,dx = \int \frac{1}{\left(\sqrt{x^2-9}\right)^2}\,dx$$

FYI

BC 2001 #5

Let f be the function satisfying $f'(x) = -3xf(x)$, for all real numbers x, with $f(1) = 4$ and $\lim_{x \to \infty} f(x) = 0$.

a) Evaluate $\int_1^{\infty} -3x f(x)\, dx$. Show the work that leads to your answer.

b) Use Euler's method, starting at $x = 1$ with a step size of 0.5, to approximate $f(2)$.

c) Write an expression for $y = f(x)$ by solving the differential equation $dy/dx = -3xy$ with the initial condition $f(1) = 4$.

What have we learned?

- What are the three cases for trig substitution?
- Can I use trig substitution to evaluate integrals where it comes in handy?