Warmup!!

Rewrite f(x) = arctanx as a power series using sigma notation

$$f(x) = \arctan x = \int \frac{1}{1+x^2}\,dx$$

$$= \int \frac{1}{1-(-x^2)}\,dx$$

$$= \int \left(\sum_{n=0}^{\infty} (-x^2)^n\right) dx$$

$$= \int \left(\sum_{n=0}^{\infty} (-1)^n x^{2n}\right) dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + C$$

$$\arctan 0 = \sum_{n=0}^{\infty} 0 + C$$

$$f(x) = \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Note, this is writing the series from 'scratch' using a/(1-r), but if we remembered

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

we could just replace the x with an x^2

9-9b: Representing Functions with Power Series

At the end of this lesson students will be able to:

- Use a known power series to construct a new power series using series operations
- Use a known power series to find the sum of a similar power series

ex) Use the series for f(x) = arctanx to approximate the value of $\int_0^{\frac{3}{4}} \arctan x^2 \, dx$ with an error ≤ 0.001

From the warmup, we know that

$$f(x) = \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

so $\arctan x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2n+1}$

$$\int_0^{\frac{3}{4}} \arctan x^2 \, dx = \int_0^{\frac{3}{4}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2n+1} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)(4n+3)} \Bigg|_0^{3/4}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{3}{4}\right)^{4n+3}}{(2n+1)(4n+3)} - \sum_{n=0}^{\infty} \frac{(-1)^n 0^{4n+3}}{(2n+1)(4n+3)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{3}{4}\right)^{4n+3}}{(2n+1)(4n+3)}$$

Because this is an alternating series, we can use the alternating series remainder theorem to find the term in the series that is ≤ 0.001.

This ends up being a_2. Be careful! This series starts with n = 0, so you will need to add a_0 + a_1 to get the correct sum (and a_1 is negative).

The final sum ≈ 0.13427

You try! Use the series for f(x) = arctanx to approximate the value of $\int_0^{\frac{1}{2}} x^2 \arctan x \, dx$ with an error ≤ 0.001.

From the warmup, we know that

$$f(x) = \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$x^2 \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{2n+1}$$

$$\int_0^{\frac{1}{2}} x^2 \arctan x \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+4}}{(2n+1)(2n+4)} \Bigg|_0^{\frac{1}{2}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)^{2n+4}}{(2n+1)(2n+4)} - \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 0}{(2n+1)(2n+4)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)^{2n+4}}{(2n+1)(2n+4)}$$

Using the calculator to check, a_1 is actually < 0.001, so the sum is just $a_0 \approx 0.01563$

ex) Use the power series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, |x| < 1$

Find the series representation of $f(x) = \frac{x(1+x)}{(1-x)^2}$

and determine the interval of convergence.

$$\frac{x(1+x)}{(1-x)^2} = x \cdot \frac{1+x}{(1-x)^2}$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} nx^{n-1}$$

$$\frac{x}{(1-x)^2} = x\sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^n$$

$$\frac{1+x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} + \sum_{n=1}^{\infty} nx^n$$

$$= 1 + \sum_{n=1}^{\infty} (n+1)x^n + \sum_{n=1}^{\infty} nx^n$$

$$= 1 + \sum_{n=1}^{\infty} (2n+1)x^n = \sum_{n=0}^{\infty} (2n+1)x^n$$

$$\frac{x(1+x)}{(1-x)^2} = \sum_{n=0}^{\infty} (2n+1)x^{n+1}$$

To find the interval of convergence:

$$\lim_{n\to\infty}\left|\frac{(2(n+1)+1)x^{n+2}}{(2n+1)x^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(2n+3)x}{2n+1}\right| = |x| < 1$$

Test the endpoints: both diverge by the nth term test so our final interval is -1 < x < 1

Let's review some 'well-known' series that we have encountered in this unit.

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$f(x) = \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$f(x) = \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$f(x) = \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$f(x) = \ln(x+1) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

$$f(x) = \frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

$$f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

Find the sum of the convergent series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ using a 'well-known' function. Identify the function and explain how you obtained the sum.

The '2n + 1' reminds me of arctanx.

$$f(x) = \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

The desired sum is the same but does not have the x. We could have an equivalent value if we evaluated the arctangent function at x = 1.

$$\arctan 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$$

What have we learned?

- Can I use a known power series to construct a new power series using series operations?
- Can I use a known power series to find the sum of a similar power series?