Warmup!!

Suppose $f(x) = \frac{4}{5-x}$

a) Rewrite f(x) in the form $\frac{a}{1-r}$

Hint (change the 5 to a 1) $f(x) = \frac{\frac{4}{5}}{1-\frac{x}{5}}$

b) Use this to write f(x) as a power series using sigma notation $\sum_{n=0}^{\infty} \frac{4}{5}\left(\frac{x}{5}\right)^n = \sum_{n=0}^{\infty} \frac{4}{5^{n+1}} x^n$

c) Now use this to write the series as a polynomial $\frac{4}{5} + \frac{4}{25}x + \frac{4}{125}x^2 + \frac{4}{525}x^3 + \cdots$

This polynomial is called the 'geometric power series' for the function, because it is based on the sum of a geometric series

$$\sum_{n=0}^{\infty} a(r)^n = \frac{a}{1-r}$$

HW 9.8b

(47)

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$$

$$f'(x)=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\cancel{(n+1)}(x-1)^{n}}{\cancel{(n+1)}}$$

$$\int f(x)\,dx=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(x-1)^{n+2}}{(n+1)(n+2)}$$

$$\lim_{n\to\infty}\left|\frac{(x-1)^{n+2}}{n+2}\cdot\frac{n+1}{(x-1)^{n+1}}\right|$$

$$\lim_{n\to\infty}\left|\frac{(x-1)(n+1)}{n+2}\right| \approx |x-1|<1$$

$R=1$ for $f(x)$, $f'(x)$, $\int f(x)\,dx$

$-1<x-1<1$

So $0<x<2$

Test $f(x)$: 0 diverges by lim comp test

2 converges by alt srs test

Test $f'(x)$: 0, 2 diverge by nth term test

Test $\int f(x)\,dx$: 0 converges by dir comp

2 converges by alt series

So:

$f(x)$: $0<x\leq 2$

$f'(x)$: $0<x<2$

$\int f(x)\,dx$: $0\leq x\leq 2$

9-9a: Representing Functions with Power Series

At the end of this lesson students will be able to:

- Find a geometric power series that represents a function
- Construct a power series using series operations

Let's go back to the original function from the warmup: $f(x) = \frac{4}{5-x}$ $= \frac{4}{5-(x+2)+\,?}$

The power series we found to represent this function would have been centered at 0. What if we wanted to represent the same function but center it at c = -2?

1) Rewrite f(x) so that the x is replaced with an x + 2

$$\frac{4}{5-x} = \frac{4}{5-(x+2)+2} = \frac{4}{7-(x+2)}$$

2) write in the form $\frac{a}{1 - r}$

$$\frac{4}{7-(x+2)} = \frac{\frac{4}{7}}{1-\frac{x+2}{7}}$$

3) Write using sigma

$$\sum_{n=0}^{\infty}\frac{4}{7}\left(\frac{x+2}{7}\right)^n = \sum_{n=0}^{\infty}\left(\frac{4}{7^{n+1}}\right)(x+2)^n$$

4) Find the radius and interval of convergence for f(x). Note that if your sigma is geometric, $\sum a(r)^n$, you can use the geometric series to determine the interval of convergence.

$$\left|\frac{x+2}{7}\right| < 1$$

so $|x+2| < 7$ and R = 7. The initial interval of convergence is $-9 < x < 5$. Testing the endpoints gives:

$$\sum_{n=0}^{\infty}\frac{4(-7)^n}{7^{n+1}} = \sum_{n=0}^{\infty}\frac{4(-1)^n}{7} \qquad \sum_{n=0}^{\infty}\frac{4(7)^n}{7^{n+1}} = \sum_{n=0}^{\infty}\frac{4}{7}$$

So the series diverges at both endpoints, so our interval of convergence is $-9 < x < 5$.

You try! Write $\frac{4}{3x+2}$ as a power series centered at x = 2. Then find the radius and interval of convergence.

Step reminder:

- Replace the 'x' with 'x - 2'.
- Convert to a/(1-r) form.
- Write with ∑.
- Find the radius/interval.

$$\frac{4}{3(x-2)+2+6} = \frac{4}{3(x-2)+8}$$

$$\frac{4}{8-\ -3(x-2)} = \frac{\frac{1}{2}}{1-\ -\frac{3}{8}(x-2)}$$

$$\sum_{n=0}^{\infty} \frac{1}{2}\left(-\frac{3}{8}(x-2)\right)^n = \sum_{n=0}^{\infty} \frac{(-3)^n}{2(8)^n}(x-2)^n$$

$$\left|\frac{-3}{8}(x-2)\right| < 1$$

So our R = 8/3 and our initial interval of convergence is -2/3 < x < 14/3. Test the endpoints to get:

$$\sum_{n=0}^{\infty} \frac{(-3)^n}{2(8)^n}\left(-\frac{2}{3}-2\right)^n = \sum_{n=0}^{\infty} \frac{1}{2}$$

diverges based on nth term test

$$\sum_{n=0}^{\infty} \frac{(-3)^n}{2(8)^n}\left(\frac{14}{3}-2\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2}$$

diverges based on nth term test

So our final interval of convergence is -2/3 < x < 14/3

You try again! (But I'll give you a nudge) 😊

Suppose $h(x) = \dfrac{x}{x^2 - 1}$. Find a power series for h(x) centered at 0. Then determine the interval of convergence.

Nudge: start by multiplying top and bottom by -1.

$$h(x) = \frac{-x}{1 - x^2}$$

So a = -x and r = x^2

$$\sum_{n=0}^{\infty}(-x)(x^2)^n = \sum_{n=0}^{\infty}(-1)(x)^{2n+1}$$

$$\lim_{n\to\infty}\left|\frac{(-1)(x)^{(2(n+1)+1)}}{(-1)(x)^{2n+1}}\right| = \lim_{n\to\infty}\left|x^{(2n+3)-(2n+1)}\right| = |x^2| < 1$$

So R = 1 and our initial interval is -1 < x < 1.

Test the endpoints to get:

$$\sum_{n=0}^{\infty}(-1)(-1)^{2n+1} = \sum_{n=0}^{\infty}(-1)^{2n+2} = \sum_{n=0}^{\infty}1$$

$$\sum_{n=0}^{\infty}(-1)(1)$$

Both diverge by the nth term test.

So our final interval is -1 < x < 1.

Use the power series $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$

to determine a power series, centered at 0, for the function, $f(x) = \ln(1 - x^2)$. Then identify the interval of convergence.

1) We need to somehow relate f(x) to the given series. Start by using properties of logs:

$$\ln(1-x^2) = \ln(1+x) + \ln(1-x)$$

Thinking about how ln(1 + x) relates to 1/(1+x) made me think of integrals, so rewrite as integrals.

$$= \int \frac{1}{1+x} dx + \int -\frac{1}{1-x} dx + C$$

Rewrite each of these as geometric power series. The first one is given in the problem.

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \qquad \frac{-1}{1-x} = -\sum_{n=0}^{\infty} (x)^n$$

So now we have:

$$\int \sum_{n=0}^{\infty} (-1)^n x^n \, dx - \int \sum_{n=0}^{\infty} x^n \, dx$$

$$\ln(1-x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C$$

To solve for C, plug in a zero for x (always use the center) to get: $\ln(1-0) = \sum_{n=0}^{\infty} 0 - \sum_{n=0}^{\infty} 0 + C$

So $C = 0$ and our series is:

$$\ln(1-x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} + \sum_{n=0}^{\infty} \frac{(-1) x^{n+1}}{n+1}$$

Note: $\sum a_n x^n \pm \sum b_n x^n = \sum (a_n \pm b_n) x^n$

So our FINAL series is:

$$\ln(1-x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n - 1}{n+1} x^{n+1}$$

Let's find that interval of convergence!

When adding 2 series together, the interval of convergence is the <u>intersection</u> of the convergence intervals for each series. (The intersection is all of the elements they have in common.) Ex: (-5, 3) ∩ [-8, 2] = (-5, 2]

Because the -1 can be dropped for the ratio test, both series have the same test:

$$\lim_{n\to\infty} \left| \frac{x^{n+2}}{n+2} \cdot \frac{n+1}{x^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{x(n+1)}{n+2} \right| = |x| < 1$$

After testing the endpoints, you will find that the intervals for both series are $-1 < x < 1$, so the final interval of the sum of the series is $-1 < x < 1$

What have we learned?

- Can I create a geometric power series based on a given function?
- Can I add/subtract 2 series together and find the interval of convergence of the sum?
- Am I a bit tired after all of this?

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