Warmup!!

Write each of the following in sigma notation. Then determine the convergence/divergence of the following series. (Hint for writing in sigma notation: each series has a polynomial in the denominator)

1) $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{35}+\cdots$ $\sum_{n=2}^{\infty}\frac{1}{n^2-1}$

converges based on the limit comparison test

2) $\frac{1}{200}+\frac{1}{210}+\frac{1}{220}+\frac{1}{230}+\cdots=\sum_{n=0}^{\infty}\frac{1}{10n+200}$

diverges based on the limit comparison test

3) $\frac{1}{301}+\frac{1}{308}+\frac{1}{327}+\frac{1}{364}+\cdots=\sum_{n=1}^{\infty}\frac{1}{300+n^3}$

converges based on the limit comparison test

4) $\frac{2}{3}+\frac{3}{8}+\frac{4}{15}+\frac{5}{24}+\frac{6}{35}+\frac{7}{48}+\cdots=\sum_{n=2}^{\infty}\frac{n}{n^2-1}$

diverges based on the limit comparison test

9-8b: Power Series Part 2

At the end of this lesson students will be able to:

- Differentiate and integrate power series
- Determine the radius and integral of convergence for the derivative and/or integral of a power series

Can we write an infinite sum as a function?

Let's give it a try!

Suppose $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ (your standard power series centered at c)

In your groups, write out the first 4 'terms' of f(x) (write f(x) without the sigma notation). Then decide if f(x) is a continuous and differentiable function.

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

Why, it's just a polynomial (which we knew, of course), so of course it's **differentiable**! Not only that, it's **integrable** as well!

Take a few minutes and find f'(x), then write f'(x) using sigma notation.

$$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$$

$$= \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$$

Now take a look at the original f(x) written in sigma notation, and think about what the derivative of this would look like. Can you make a conjecture based on this on how to differentiate a power series?

Let's work in the other direction.

Remember that:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

$$= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

Use the polynomial version of f(x) to find ∫f(x)dx.

Then write the answer using sigma notation.

$$\int f(x)dx = C + a_0(x-c) + \frac{1}{2}a_1(x-c)^2 + \frac{1}{3}a_2(x-c)^3 + \cdots$$

$$= C + \sum_{n=0}^{\infty} \frac{1}{n+1} a_n (x-c)^{n+1}$$

Is it 'okay' to have an 'x - c' after the a_0 term, rather than just an x? Why or why not?

Mentally integrate the original sigma function and determine if you obtain the same answer. Can you make a conjecture about the integral of a power series?

ex) Suppose $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$

Determine the radius and interval of convergence for f(x).

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right| = \lim_{n\to\infty} \left| \frac{xn}{n+1} \right| = |x| < 1$$

So R = 1 and our starting interval is $-1 < x < 1$

Testing x = -1 we get: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

which converges based on the alternating series test.

Testing x = 1 we get: $\sum_{n=1}^{\infty} \frac{1}{n}$

which diverges based on the p-test

So our final interval of convergence is $-1 \leq x < 1$

For the same function: $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$

Find f'(x) and determine the radius and interval of convergence for f'(x).

$$f'(x) = \sum_{n=1}^{\infty} x^{n-1}$$

$$\lim_{n\to\infty} \left| \frac{x^n}{x^{n-1}} \right| = \lim_{n\to\infty} |x| = |x| < 1$$

So R = 1 and our interval starts with -1 < x < 1

Testing x = -1 we get: $\sum_{n=1}^{\infty} (-1)^{n-1}$

Testing x = 1 we get: $\sum_{n=1}^{\infty} 1$

Both of these diverge based on the nth term test so our interval of convergence is -1 < x < 1

For the same function: $f(x)=\sum_{n=1}^{\infty}\frac{x^n}{n}$

Find ∫f(x)dx and determine the radius and interval of convergence for ∫f(x)dx.

$$\int f(x)dx = C + \sum_{n=1}^{\infty}\frac{x^{n+1}}{n(n+1)}$$

$$\lim_{n\to\infty}\left|\frac{\frac{x^{n+2}}{(n+1)(n+2)}}{\frac{x^{n+1}}{n(n+1)}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^{n+2}}{(n+1)(n+2)}\cdot\frac{n(n+1)}{x^{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{xn}{n+2}\right| = |x| < 1$$

So R = 1 and our starting interval is $-1 < x < 1$

Test x = -1: $\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+1)}$ Test x = 1: $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$

So the sum converges at x = -1 based on the alternating series test, but converges at x = 1 based on the direct comparison.

Our interval of convergence is $-1 \le x \le 1$

Looking at the last example, compare the radius and interval of convergence for f, f'(x) and $\int f(x)dx$. Do you see any similarities or differences? What conclusions do you think you can draw?

From the example:

f(x): R = 1 convergence interval: $-1 \le x < 1$

f'(x): R = 1 convergence interval: $-1 < x < 1$

$\int f(x)dx$: R = 1 convergence interval: $-1 \le x \le 1$

- The radius of convergence will always be the same for both the derivative and integral of a power series. ✓
- The open interval of convergence will always be the same for both the derivative and integral of a power series. However, the final intervals might differ at the endpoints.

So knowing this, would you still have to use the limit comparison test on $f'(x)$ and $\int f(x)dx$ to find the radius and interval of convergence?

Describe the minimum steps you would need in order to find the radius and interval of convergence for all three.

- Use the ratio test to find the radius and open interval of convergence for $f(x)$
- Find $f'(x)$ and $\int f(x)dx$
- Test the endpoints for all three to find the final intervals of convergence

Let's just make sure you've got it.

Find the radius and intervals of convergence for f(x), f'(x) and ∫f(x)dx if:

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n}$$

$$\lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{n+1}}{\frac{(x-2)^n}{n}} \right| = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{n+1} \cdot \frac{n}{(x-2)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x-2)n}{n+1} \right| = |x-2| < 1$$

So the radius of convergence for all three is 1 and the initial interval to check is $1 < x < 3$.

$$f'(x) = \sum_{n=1}^{\infty} (-1)^{n+1}(x-2)^{n-1}$$

$$\int f(x)dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^{n+1}}{n(n+1)}$$

Test the endpoints for f(x):
1 diverges based on the p-test
3 converges based on the alternating series test
So the interval of convergence for f(x) is $1 < x \le 3$

Test the endpoints for f'(x)
1 diverges based on the nth term test
3 diverges based on the nth term test
So the interval of convergence for f'(x) is $1 < x < 3$

Test the endpoints for ∫f(x)dx
converges at 1 based on the direct comparison test
converges at 3 based on the alternating series test
So the interval of convergence for ∫f(x)dx is $1 \le x \le 3$

Srinivasa Ramanujan!

Srinivasa was born in 1887 and grew up in a town not terribly far from Madras, India. He did well in school and after graduating from high school chose to study mathematics on his own, at first simply using a book he found in his high school. In 1902 he developed a method for solving a quartic polynomial equation. by 1904 he was focusing his research on infinite series, and independently discovered Bernoulli numbers without knowing they had already been discovered by Bernoulli. Because of his work, he was granted a scholarship to the Government College of Kumbakonam, but the scholarship was not renewed his 2nd year because he ignored all other college subjects besides math.

Srinivasa 'ran away' to a new town and focused independently on hypergeometric series, where he again independently discovered elliptic functions without knowing they had already been discovered. In 1906 he tried to enter the University of Madras but failed the entrance exams for all subjects besides math. So he decided to focus instead on studying continued fractions and divergent series. In 1909 his mother arranged for him to marry a 10-year old girl, but they didn't move in together until she was 12. In the meantime in 1911 he published a highly acclaimed paper on Bernoulli numbers in an Indian mathematical journal. This finally gave him some recognition and he became known as a mathematical genius. Because of this the founder of the Indian Mathematical Society arranged for him to get his first job (accounts clerk at the Madras Port Trust). This is what he wrote when Ramanujan walked in:

> *A short uncouth figure, stout, unshaven, not over clean, with one conspicuous feature-shining eyes- walked in with a frayed notebook under his arm. He was miserably poor. ... He opened his book and began to explain some of his discoveries. I saw quite at once that there was something out of the way; but my knowledge did not permit me to judge whether he talked sense or nonsense. ... I asked him what he wanted. He said he wanted a pittance to live on so that he might pursue his researches.*

In 1913, Ramanujan wrote to G H Hardy, who had written a book on orders of infinity, and Hardy was very impressed with his work. Through Hardy, Ramanujan was quickly admitted to the Univesity of Madras, and then quickly transferred to Trinity College in London. Unfortunately Ramanujan was seriously ill for many of his adult years and eventually moved back to India to try to recover but died very young in 1920.

While in London, Ramanujan became famous for his mathematical contributions. His only major setback was that because he had so little formal training he did not have a full grasp of how to write a proof and some of his theorems were completely wrong. But it was later shown that he had independently discovered Riemann series, elliptic integrals, hypergeometric series, and many of the results of Gauss, Kummer, and others. He created a series for approximating $1/\pi$ where each successive term adds 8 more correct digits to the approximation.

What have we learned?

- Can a power series be written as a function? Why?
- Can I differentiate and integrate a power series?
- Can I determine the radius and interval of convergence of the derivative and/or integral of a power series?

Gee whiz, it's time for a quiz!!