Warmup!!

Write the first 4 terms of the Maclaurin series for each function below. Then rewrite the series using sigma notation.

1) $f(x) = \dfrac{1}{1-x}$ ✓

$$P(x) = 1 + x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 + \cdots$$

$$= 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

2) $f(x) = e^x$ ✓

$$P(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

3) $f(x) = \cos x$

✓

$$P(x) = 1 + 0x + \frac{-1}{2!}x^2 + 0x^3 + \frac{1}{4!}x^4 + 0x^5 + \frac{-1}{6!}x^6 + \cdots$$

$$= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot x^{2n}$$

HW

(57) $f(x) = e^x$

$$\approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$\text{error} = |R| < .001$

$$|R| < \left| \frac{(x-c)^{n+1}}{(n+1)!} \max f^{(n+1)}(z) \right|$$

$c = 0$

$n = 3$

$$\left| \frac{x^4}{4!} \left(\begin{matrix} \text{max of} \\ e^x \text{ between } x \text{ and } c \end{matrix} \right) \right| < .001$$

$$|x^4 (1)| < .024$$

$e^0 = 1$

$e^{-\#} = \frac{1}{e^{\#}}$

$$|x| < \sqrt[4]{.024}$$

$$|x| < .393979$$

$$-.393979 < x < 0$$

(53)

$f(x) = \ln(x+1)$

$f(0.5)$

$p(x) \approx 0 + \frac{x}{2} + \frac{-x^2}{18} + \frac{x^3}{192} + \ldots$

$x = .5$
$c = 0$ $R < .0001$

$R < \left| \frac{(.5)^{n+1}}{(n+1)!} \text{ max of } n+1^{th} \text{ deriv on } [0, .5] \right| < .0001$

to find this requires the derivative series which you won't learn until section 9-8b, so for this problem I would just plug 0.5 into the Maclaurin until you are within .0001 of the actual value :)

9-8a: Power Series Part 1

At the end of this lesson students will be able to:

- Understand what a power series is
- Find the radius and interval of convergence of a power series

What is a power series?

A power series (centered at c) is any series of the form:

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

A power series centered at 0 would be:

$$\sum_{n=0}^{\infty} a_n x^n$$

So, all of the series we found in the warmup would be considered power series - even the cosx series.

2 Important Definitions:

RADIUS OF CONVERGENCE

There exists a number $R > 0$ such that the series converges absolutely for $|x - c| < R$ and diverges for $|x - c| > R$. R is called the radius of convergence of the power series.

INTERVAL OF CONVERGENCE

The set of all values of x for which the power series converges.

$a - R < x < a + R$	power series converges
$x < a - R$ and $x > a + R$	power series diverges

ex) Find the radius and interval of convergence of $\sum_{n=0}^{\infty} 3(x-2)^n$

The best test to use for determining R is the ratio test.

We know by looking at the series that c = 2.

Using the ratio test we get:

$$\lim_{n\to\infty}\left|\frac{3(x-2)^{n+1}}{3(x-2)^n}\right| = \lim_{n\to\infty}|x-2| = |x-2|$$

For the series to converge, $|x - 2| < 1$, so R = 1.

Based on this, the interval of convergence might seem simple to compute.

$-1 < x - 2 < 1$ so $1 < x < 3$

However, we need to determine if convergence also occurs at the endpoints of this interval or not. We will need to check each endpoint individually and check for convergence.

If x = 1, we get $\sum_{n=0}^{\infty} 3(-1)^n$

which diverges based on the alternating series test. So x = 1 would not be in the interval.

If x = 3, we get $\sum_{n=0}^{\infty} 3(1)^n$

which also diverges based on the nth term test. So x = 3 is also not in the interval.

SO, R = 1 and the interval of convergence is $1 < x < 3$.

You try! Find the radius and interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n}$

We know by looking at the series that c = 0.

Using the ratio test we get:

$$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right| = \lim_{n\to\infty}\left|\frac{xn}{n+1}\right| = |x|$$

For the series to converge, $|x| < 1$, so R = 1.

Based on this, the interval of convergence starts with $-1 < x < 1$, but we need to check the endpoints.

If x = -1, we get $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$

which converges based on the alternating series test. So x = -1 would be in the interval.

If x = 1, we get $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n}$

which also diverges based on the p-test. So x = 1 is not in the interval.

SO, R = 1 and the interval of convergence is $-1 \leq x < 1$.

One more! Find the radius and interval of convergence for the series: $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

$$\lim_{n\to\infty} \left| \frac{\frac{x^{2(n+1)+1}}{2(n+1)+1}}{\frac{x^{2n+1}}{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^{2n+3}}{2n+3} \cdot \frac{2n+1}{x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^2(2n+1)}{2n+3} \right| = |x^2| = x^2 < 1$$

The $(-1)^n$ term does not have to be included when applying the ratio test because everything is in an absolute value 😊

So we end up with $-1 < x < 1$ and $R = 1$

Testing at $x = -1$ we get:

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(-1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{3n+2}}{2n+1}$$

Which converges based on the alternating series test.

Testing at $x = 1$ we get: $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$

Which also converges based on the alternating series test.

So the interval of convergence is $-1 \le x \le 1$.

AP Practice!!

1985 #31

What are all x-values for which the series $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$ converges?

a) $-1 \le x < 1$
b) $-1 \le x \le 1$
c) $0 < x < 2$
d) $0 \le x < 2$
e) $0 \le x \le 2$

answer is d!

1997 #20

What are the values of x for which the series converges?

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n \cdot 3^n}$$

a) $-3 \le x \le 3$
b) $-3 < x < 3$
c) $-1 < x \le 5$
d) $-1 \le x \le 5$
e) $-1 \le x < 5$

answer is e!

1997 #20 worked out

$$\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)\cdot 3^{n+1}}\cdot\frac{n3^n}{(x-2)^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(x-2)n}{(n+1)3}\right|<1$$

$$=\lim_{n\to\infty}\left|\frac{(x-2)n}{3n+3}\right|<1$$

$$\left|\frac{x-2}{3}\right|<1$$

$$|x-2|<3$$

$$-3<x-2<3$$

$$-1\le x<5$$

$$\sum_{n=1}^{\infty}\frac{(-3)^n}{n\cdot 3^n}=\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 3^n}{n3^n}$$

$$\sum_{n=1}^{\infty}\frac{3^n}{n\cdot 3^n}\quad \text{div}$$

What have we learned?

- What is a power series?
- Which test is usually the best one to use to determine the radius and interval of convergence?
- What do I need to make sure to check when writing the interval of convergence?