Warmup!!

Suppose $f(x)$ is represented by the following Taylor Polynomial:

$$P(x) = 4 - 5(x-3) + 3(x-3)^2 - \frac{7}{6}(x-3)^3 + \frac{1}{3}(x-3)^4$$

What is the value of $f'''(3)$?

Since $\dfrac{f'''(3)}{3!} = -\dfrac{7}{6}$, $f'''(3) = -7$

9-7b: Taylor Polynomials Part 2

Essential Learning Targets:

- I can use the Lagrange error bound to bound the error of a Taylor approximation to a function
- In some situations where the signs of the Taylor polynomial are alternating, I can use the alternating series error bound to bound the error of a Taylor approximation to a function

Finding the Taylor approximation for a function value at a specific point is just like finding the linear approximation for a function value at a specific point.

LINEAR APPROXIMATION

- Find the equation of the tangent line
- Plug in the given value

TAYLOR APPROXIMATION

- Find the Taylor polynomial
- Plug in the given value

You try! Use a fourth degree Maclaurin polynomial to approximate the value of $e^{(0.2)}$. (Hint: you can save some time looking through yesterday's notes.) (calculators permitted)

From yesterday we learned that

✓ $P_4(x) = 1 + x + 1/2\ x^2 + 1/6\ x^3 + 1/24\ x^4$

so $P_4(0.2) = 1 + 0.2 + 0.04/2 + 0.008/6 + 0.0016/24$

$= 1.2214$

Name 2 ways we can improve an approximation:

- make the x-value closer to the c-value (or vice versa)
- use a higher-degree polynomial

But how do we know if we need improving?
How accurate are these approximations?

Think of an actual value as an approximate value + a remainder

$f(x) = P_n(x) + R_n(x)$

so ERROR = $|R_n(x)| = |f(x) - P_n(x)|$

<u>LaGrange Error Bound</u> (in your book it's called Taylor's Theorem)

If f is differentiable through order n + 1 in an interval containing c, then for each x in the interval there exists z between x and c such that

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$

can't be found ha ha!

Because of this, the remainder will always be:

$$0 < |R_n(x)| \leq \frac{(x-c)^{n+1}}{(n+1)!} \cdot max|f^{n+1}(z)|$$

in $[c, x]$

x = x-value where we are approximating the function
c = center of polynomial
n = degree of polynomial
z = any number between x and c
(z will never be known, so to find the max value of the derivative, just find the derivative at x, the derivative at c and take the higher value. For sine and cosine, let the max = 1)

Let's find the error in our approximation of $e^{0.2}$!

$c = 0$

$x = 0.2$

$n = 4$

$(n+1 = 5)$

$$\text{error} \le \frac{(0.2-0)^5}{5!}\left(\begin{array}{l}\text{max value of}\\ 5^{\text{th}}\text{ deriv of}\\ e^x \text{ on } [0, 0.2]\end{array}\right)$$

$\approx .000003257$

ex) A Maclaurin polynomial for $f(x) = \sin x$ is given by $P(x) = x - x^3/6$.

Approximate sin(0.1) and use the Lagrange error bound to determine the accuracy of the approximation.

$P(0.1) = 0.1 - 0.001/6 \approx 0.09983333$

even though this is a 3rd degree, we are going to assume it's a 4th degree with a zero term and let the 'next' term be the 5th degree

$x = 0.1$ $\quad c = 0$ $\quad n = 4$ so $n + 1 = 5$

$$0 < |R_n(x)| \le \frac{(x-c)^{n+1}}{(n+1)!} \cdot max|f^{n+1}(z)|$$

So $\text{Error} \le \frac{(.1-0)^5}{5!} \cdot \max|f^{(5)}(z)|$ on $[0, .1]$

$= \frac{.000001}{120}(1) \approx .00000008333$

You try! Determine an upper-bound for the error of the approximation if

$$e \approx 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} + \frac{1^5}{5!}$$

$c=0$
$x=1$
$n=5$

Then calculate the exact error.

- Start by finding x and c (if you can't figure these out, write out the Taylor Polynomial for e^x centered at 0)

$$e^x \approx 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

So x = 1 and c = 0. Looking at the given problem, n = 5.

$$R_5(1) \le \frac{(1-0)^6}{6!} \cdot \max\ |6th\ derivative\ of\ e^x on\ [0,1]|$$

$$= \frac{e^1}{720} \approx 0.0037754$$

So our upper bound is 0.0037754

Exact error:

Since the Taylor polynomial approximation $\frac{163}{60}$

The exact error $= e - \frac{163}{60} \approx 0.0016152$

exact approx actual error

Brain Break!!

Mathematician Monday!!

Katherine Okikiolu!

Katherine was born in 1965 in England. Her father, George, was Nigerian but moved to England to study mathematics where he met her mother who was studying physics. (BTW, her father is the most published black mathematician in history.)

Katherine earned her BA from Cambridge and her PhD from UCLA. While at UCLA she proved her abilities by solving a problem involving asymptotics of determinants of Toeplitz operators on the sphere as well as a conjecture of Peter Jones, characterizing subsets of rectifiable curves in Euclidean n-space.

Not only was Katherine the first black mathematician to earn the Sloan Research Fellowship, she was also awarded a Presidential Early Career Award for Scientists and Engineers ($500,000) for both her mathematical research and her development of a math curriculum for inner-city school children. She created a method for teaching negative numbers using money and debt and is working on extending the lessons to a large number of real-world applications in order to teach simple mathematical concepts.

Katherine has taught at Princeton, MIT, and UC San Diego, and is currently at Johns Hopkins. One of her areas of study has been on drums. She has been researching the "spectral determinant" of a drum, which is essentially the number obtained by multiplying all the individual sound pitches made from a drum note. This number helps describe the shape of the drum. This has been done already for 2-dimensional drums, but Katherine is extending the study to 3-dimensions.

Other major contributions of note: Katherine's work in elliptical differential operators is considered a major contribution, going well beyond what experts had considered feasible given the current state of knowledge. She was the first black female to publish an article in The Annals of Mathematics. Her article was called "Critical metrics for the determinant of the Laplacian in odd dimensions." Her "Characterization of subsets of rectifiable curves in $\mathbf{R}^n$", is one of the few foundations of the Fractal Instances of the Traveling Sales Problem.

You try!

Determine the degree of the Maclaurin polynomial required for the error in the approximation to be less than 0.0001 if $f(x) = e^{-x}$ and we are approximating f(1).

- Find x and c

$$|R_n(1)| \leq \left|\frac{(1-0)^{n+1}}{(n+1)!} \cdot \max|(n+1)th\ derivative\ of\ e^{-x}\ on\ [0,1]|\right| < .0001$$

- the (n+1)th derivative of e^{-x} is e^{-x} (we only need the absolute value)
- so the max of e^{-x} on [0, 1] is 1

So we have $\dfrac{1}{(n+1)!} < 0.0001$

Solving we get: $(n+1)! > 10000$

By trial and error we get that 8! > 10000 so n = 7

If you are really stuck, sometimes you can 'cheat' the system (only on certain types of problems).

ex) Find a Maclaurin polynomial approximation for y = cos(0.1) that is within 0.0000001 of the actual value.

$f(x) = \cos x$ $f(0) = 1$

$f'(x) = -\sin x$ $f'(0) = 0$

$f''(x) = -\cos x$ $f''(0) = -1$

$f'''(x) = \sin x$ $f'''(0) = 0$

$f''''(x) = \cos x$ $f''''(0) = 1$

So $P(x) = 1 - x^2/2! + x^4/4! - x^6/6! + x^8/8! - \ldots$

We know the actual value of cos(0.1) is

$\cos(0.1) \approx 0.995004165278$

Start plugging 0.1 into the polynomial until you reach a value that is within 0.0000001 of this.

$P_0(0.1) = 1$

$P_2(0.1) = 1 - (.1)^2/2 = 0.995$

$P_4(0.1) = 0.995 + (.1)^4/24 = 0.99500416666$

bingo!

What have we learned?

- Can I approximate the remainder for a Taylor polynomial?
- Can I determine values of n to create a polynomial with a remainder less than a given value?
- Can I determine values of x for which a given polynomial will have a remainder less than a given value?