Warmup!!

Let's review with 2012 #1

(calculators permitted)

9-7a: Taylor Polynomials Part 1

Essential Learning Targets:

- I can calculate the coefficients for Taylor polynomials
- I understand that Taylor polynomials for a function centered at x = a can be used to approximate functions values near x = a
- I understand that in many cases, as the degree of the Taylor polynomial increases, the nth degree polynomial will converge to the original function over some interval

We love polynomials!

Polynomials are the easiest functions to differentiate, integrate, graph, and analyze. Don't you just love to sit and ponder how great life would be if all functions were polynomials?

In this section, we will learn how to turn any function into a polynomial!

What!?! How is this possible?

Let's see!!

Don't write, just sit back and digest

To create a polynomial approximation you need to be given 2 things:

- the center of the polynomial (a point that is shared by the original function and the polynomial)
- the 'accuracy' of the polynomial

For example: Create a first-degree polynomial to approximate $f(x) = e^x$ whose value and slope agree with f at $x = 0$.

A first-degree polynomial looks like $p(x) = mx + b$, but we will use $p(x) = a_0 + a_1x$

To make the point agree we get:

Since $f(0) = 1$, we know $p(0) = a_0 = 1$.

To make the slope agree we get:

Since $f'(x) = e^x$ and $f'(0) = 1$, we know $a_1 = 1$.

So our polynomial is $p(x) = x + 1$.

This is all well and good, but how can a line approximate a curvy function? Can we improve it? Yes we can!

To improve the approximation, you can impose an additional condition that the second derivative must also agree at $x = 0$.

Since $f''(x) = e^x$ and $f''(0) = 1$, we know that $p''(0)$ must also equal 1. For this to happen, we need to make p(x) a second-degree polynomial.

So now $p(x) = a_0 + a_1x + a_2x^2$.

Since $p'(x) = a_1 + 2a_2x$ and $p''(x) = 2a_2$,

so $2a_2 = 1$ and $a_2 = 1/2$.

Now our polynomial is $p(x) = {}^1/_2x^2 + x + 1$.

You try! Continue the same example and find a third-degree polynomial approximation for $f(x) = e^x$ where the first three derivatives all equal at $x = 0$. We already know a_0, a_1, and a_2 from our previous problem. To find a_3, find $f'''(x)$ at $x = 0$. Then find $p'''(x)$, set them equal to each other and solve.

✓ $p(x) = 1/6\ x^3 + 1/2\ x^2 + x + 1$

If we were to continue this exercise, we would end up with the following polynomial:

$p(x) = 1 + x + 1/2\ x^2 + 1/6\ x^3 + 1/24\ x^4 + 1/120\ x^5 + \ldots$

Do you see a pattern with the coefficients?

They are all factorials!!!

How does our polynomial look? Let's see!

If you take this pattern and expand it to all types of functions (not just functions where the derivative is 1 every time), you get this:

If f has n derivatives at c, then the nth Taylor polynomial is:

center

nth derivative of f

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

If c = 0, then the nth Maclaurin polynomial is:

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

ex) Find the 4th degree Taylor polynomial for $f(x) = \ln x$ centered at $c = 1$.

It's usually helpful to start by finding the function and its first 4 derivatives at $x = 1$:

$f(x) = \ln x$, so $f(1) = 0$

$f'(x) = 1/x$, so $f'(1) = 1$

$f''(x) = -1/x^2$, so $f''(1) = -1$

$f'''(x) = 2/x^3$, so $f'''(1) = 2$

$f''''(x) = -6/x^4$, so $f''''(1) = -6$

So our 4th Taylor polynomial approximation is:

$$P_4(x) = 0 + 1(x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3 - \frac{6}{24}(x-1)^4$$

$f'(1)$ $f''(1)$ $f'''(1)$ $f''''(1)$ $1!$ $2!$ $3!$ $4!$

$$P_4(x) = x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$

You try! Find the 3rd Taylor polynomial for $f(x) = \sqrt[3]{x}$ centered at c = 8.
(calculators permitted)

$$f(x) = \sqrt[3]{x}$$

$$f(8) = \sqrt[3]{8} = 2$$

$$f'(x) = \frac{1}{3\sqrt[3]{x^2}} \; so \; f'(8) = \frac{1}{12}$$

$$f''(x) = \frac{-2}{9\sqrt[3]{x^5}} \; so \; f''(8) = -\frac{1}{144}$$

$$f'''(x) = \frac{10}{27\sqrt[3]{x^8}} \; so \; f'''(8) = \frac{5}{3456}$$

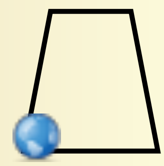
$$P_3(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{144}\cdot\frac{1}{2}(x-8)^2 + \frac{5}{3456}\cdot\frac{1}{6}(x-8)^3$$

$$P_3(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20736}(x-8)^3$$

✓

Brain Break: It's Fun Friday!!

All right, I know this one is recent, but my son was studying Rube Goldberg machines and I thought this would be fun to watch.



Maclaurin polynomials are even easier because c is automatically 0!

You try! Find the 3rd Maclaurin polynomial for f(x) = sin(πx)

$$f(x) = \sin(\pi x)$$

$$f(0) = 0$$

$$f'(x) = \pi\cos(\pi x) \; so \; f'(0) = \pi$$

$$f''(x) = -\pi^2 \sin(\pi x) \; so \; f''(0) = 0$$

$$f'''(x) = -\pi^3 \cos(\pi x) \;\; so \; f'''(0) = -\pi^3$$

$$P_3(x) = 0 + \pi x + 0x^2 - \pi^3 \cdot \frac{1}{6}x^3$$

$$P_3(x) = \pi x - \frac{\pi^3}{6}x^3$$

What have we learned?

- Can I write a Taylor polynomial?
- Can I write a Maclaurin polynomial?