Warmup!!

Determine convergence or divergence for each of the following series. When possible, find the sum.

1) $\sum_{n=1}^{\infty} \frac{5 \cdot 6^{n+2}}{3^{n-1} \cdot 2^{2n}}$

$6^n \cdot 6^2$

$$\sum_{n=1}^{\infty} \frac{5 \cdot 6^n \cdot 6^2}{3^n \cdot 3^{-1} \cdot 4^n} = \sum_{n=1}^{\infty} \frac{180}{\frac{1}{3}} \left(\frac{6}{3 \cdot 4}\right)^n = \sum_{n=1}^{\infty} 540 \left(\frac{1}{2}\right)^n$$

✓

$$= \sum_{n=0}^{\infty} 270 \left(\frac{1}{2}\right)^n = \frac{270}{1 - \frac{1}{2}} = 540$$

2) $\sum_{n=1}^{\infty} \frac{4e^n - n^3}{9e^n + 2n^2}$

Hmmm, let's check the limit as n ➔ ∞

✓

$$\lim_{n\to\infty} \frac{4e^n - n^3}{9e^n + 2n^2} = \lim_{n\to\infty} \frac{4e^n - 3n^2}{9e^n + 4n} = \lim_{n\to\infty} \frac{4e^n - 6n}{9e^n + 4}$$

$$= \lim_{n\to\infty} \frac{4e^n - 6}{9e^n} = \lim_{n\to\infty} \frac{4e^n}{9e^n} = \frac{4}{9} \neq 0$$

So by the nth term test, the series diverges

9-4: Series Comparisons!

At the end of this lesson you will be able to:

- Use the direct comparison test and the limit comparison test to determine convergence/divergence of a series

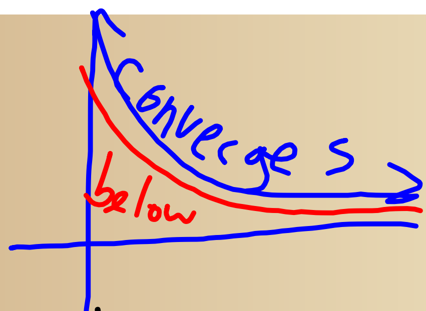
Direct Comparison Test

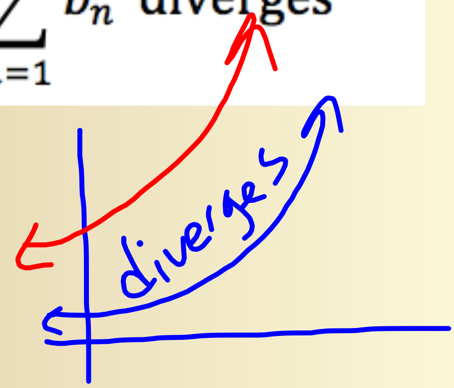
Suppose $0 < a_n \leq b_n$ for all n.

If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges

Note:





smaller than converging function converges

larger than a diverging function diverges

$\frac{1}{5}, \frac{1}{11}, \frac{1}{29}, \frac{1}{83}, \ldots$

converges

ex) $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$

Since this closely resembles $\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum \left(\frac{1}{3}\right)^n$

let's compare these two.

We know that $\frac{1}{2+3^n} < \frac{1}{3^n}$ for all n.

Since $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges (geometric with $|r| < 1$),

$\sum_{n=1}^{\infty} \frac{1}{2+3^n}$ must also converge.

You try!

$$\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$$

This converges.

Compare to $1/(3n^2)$ which converges based on the p-test.

$$\sum_{n=1}^{\infty} \frac{1}{3n^2} \leftarrow \text{converges}$$

$$\frac{1}{3n^2+2} < \frac{1}{3n^2}$$

$\therefore$ converges

Sometimes the test doesn't work for the first few terms. That's okay! If the test works for all terms from a constant value of n to infinity it can still be applied. Also, if your first comparison function doesn't work, try another one!

You try with this one! $\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$ $\not> \frac{1}{\sqrt{n}}$?

$\frac{1}{\sqrt{n}}$ After looking at the terms, these terms will always be larger than the original series, but because this series diverges we can't use this to make any conclusions

n	1	2	3	4	5	6	7	8	9
$\frac{1}{2+\sqrt{n}}$	0.333	0.293	0.268	0.25	0.236	0.225	0.215	0.207	0.2
$\frac{1}{\sqrt{n}}$	1	0.707	0.577	0.5	0.447	0.408	0.377	0.354	0.333
$\frac{1}{n}$	1	0.5	0.333	0.25	0.2	0.167	0.143	0.125	0.111

After eliminating $\frac{1}{\sqrt{n}}$ I tried 1/n. The terms of this series are larger than the original series up to n = 4, but once you get past the 4th term, the terms are all smaller and will continue in this manner up to infinity. Because the series 1/n diverges (p-test), this implies that the original series diverges as well.

Limit Comparison Test!

Suppose that $a_n > 0$ and $b_n > 0$.

If $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right)$ is both finite and positive,

then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

ex) Suppose you want to determine if $\sum_{n=1}^{\infty} \frac{1}{3n+2}$ converges or diverges

Compare this to $\sum_{n=1}^{\infty} \frac{1}{n}$

By the p-test, we know the second series diverges. Since all of the terms of the first series are less than the second series, the direct comparison test doesn't give us any information. Let's try the limit comparison test!

$$\lim_{n\to\infty} \frac{\frac{1}{3n+2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{3n+2} = \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3}$$

So the limit is finite and positive. Since the second series diverges, this implies that the first series diverges as well.

You try! Determine the convergence/divergence of

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$$

Compare this with $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$

By the p-test, we know this series converges.

Let's check the limit!

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n^2+1}}{\frac{\sqrt{n}}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$$

Because this limit is finite and positive, and because the second series converges, the first series must converge as well.

You try! (Let's take it up a notch)

$$\sum_{n=1}^{\infty} \frac{n \cdot 2^n}{4n^3 + 1}$$

Compare this with $\sum_{n=1}^{\infty} \frac{n \cdot 2^n}{n^3} = \sum_{n=1}^{\infty} \frac{2^n}{n^2}$

By the nth term test, we know this series diverges. Let's check the limit!

$$\lim_{n \to \infty} \frac{\frac{n \cdot 2^n}{4n^3 + 1}}{\frac{2^n}{n^2}} = \lim_{n \to \infty} \frac{n^3}{4n^3 + 1} = \frac{1}{4}$$

Because this limit is finite and positive, and because the second series diverges, the first series must diverge as well.

Let's summarize the tests we've learned so far:

SEQUENCES

A sequence is <u>convergent</u> if the limit exists.
A sequence is <u>divergent</u> is the limit does not exist.

Any sequence that is both monotonic and bounded will converge.

SERIES

Nth Term Test for Divergence

$$\text{If } \lim_{n\to\infty} a_n \neq 0, \text{ then } \sum_{n=1}^{\infty} a_n \text{ diverges}$$

Geometric Series Convergence

If the series is of the form $\sum_{n=0}^{\infty} a \cdot r^n$ then the series will converge to $\frac{a}{1-r}$ as long as $|r| < 1$.

Telescoping Series

$$\sum_{n=1}^{\infty} a_n - b_n = a_1 - \lim_{n\to\infty} b_n$$

Integral Test for Convergence

If f is <u>positive</u>, <u>continuous</u>, and <u>decreasing</u> for $x \geq 1$

and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad and \quad \int_1^{\infty} f(x)dx$$

either BOTH CONVERGE or BOTH DIVERGE

P-Test for Convergence

The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

converges if $p > 1$ and diverges if $0 < p \leq 1$

Direct Comparison Test for Convergence

Suppose $0 < a_n \leq b_n$ for all n.

$$\text{If } \sum_{n=1}^{\infty} b_n \text{ converges, then } \sum_{n=1}^{\infty} a_n \text{ converges}$$

$$\text{If } \sum_{n=1}^{\infty} a_n \text{ diverges, then } \sum_{n=1}^{\infty} b_n \text{ diverges}$$

Limit Comparison Test for Convergence

Suppose that $a_n > 0$ and $b_n > 0$.

If $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right)$ is both finite and positive,

then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Determine convergence or divergence for each of the following series. When possible, find the sum.

1) $$\sum_{n=2}^{\infty} \frac{3n+2}{n^2} =$$

$$\sum_{n=2}^{\infty} \frac{3n+2}{n^2} = \frac{8}{4} + \frac{11}{9} + \frac{14}{16} + \frac{17}{25} + \cdots$$

Compare this with

$$\sum_{n=2}^{\infty} \frac{3n}{n^2} = \frac{6}{4} + \frac{9}{9} + \frac{12}{16} + \frac{15}{25} + \cdots$$

✓

So the terms of the second series are always less than the terms of the first series. Since the second series diverges based on the p-test, the first series must also diverge based on the direct comparison test. (Of course you could also use the limit comparison test for this one.)

2) $$\sum_{n=1}^{\infty} \frac{6 \cdot 3^{n+2}}{5^{n+1} \cdot 2^{n+3}}$$

$$= \sum_{n=1}^{\infty} 6 \cdot \frac{3^n \cdot 3^2}{5^n \cdot 5 \cdot 2^n \cdot 2^3}$$

$$= \sum_{n=1}^{\infty} \frac{27}{20}\left(\frac{3}{10}\right)^n = \sum_{n=0}^{\infty} \frac{81}{200}\left(\frac{3}{10}\right)^n$$

✓

Geometric: converges to

$$\frac{\frac{81}{200}}{1-\frac{3}{10}} = \frac{81}{200} \cdot \frac{10}{7} = \frac{81}{140}$$

3) $\sum_{n=3}^{\infty} \frac{6}{n^2+2n}$

✓

$$= \sum_{n=3}^{\infty}\left(\frac{3}{n} - \frac{3}{n+2}\right) = 1 - \frac{3}{5} + \frac{3}{4} - \frac{3}{6} + \frac{3}{5} - \frac{3}{7} + \frac{3}{6} - \frac{3}{8} + \cdots$$

$$= 1 + \frac{3}{4} - \lim_{n\to\infty} \frac{3}{n+2} = \frac{7}{4} - 0 = \frac{7}{4}$$

4) $\sum_{n=3}^{\infty} \frac{3}{1+n^2}$

Compare this with $\sum_{n=3}^{\infty} \frac{3}{n^2}$

$$\lim_{n\to\infty} \frac{\frac{3}{1+n^2}}{\frac{3}{n^2}} = \lim_{n\to\infty} \frac{n^2}{1+n^2} = 1$$

✓

Because $3/n^2$ converges based on the p-test, the original series must also converge based on the limit comparison test

What have we learned?

- Can we apply the limit comparison and direct comparison tests to determine convergence/divergence?
- Can I recognize which test to use for a given series?