February 26, 2016

Warmup!

#1 Put the following families of functions in order based on growth rates (slower growing functions to faster growing functions)

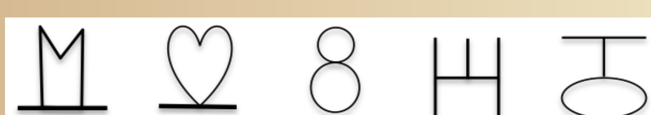
exponential
logarithmic ✓ logarithmic, polynomial, exponential, factorial
factorial
polynomial (lump in x to any positive constant exponent, including roots)

#2 Write the next 2 terms for each of the following sequences:

a) 2, 3, 5, 7, 11, 13, ... 17, 19 (primes)

b) 77, 49, 36, 18, ... ✓ 8 (that's it :))

c) O, T, T, F, F, S, S, ... E, N (number names)

d)  

9-1b: More Limits of Sequences!

At the end of this lesson you will be able to:

- determine if a sequence converges or diverges
- write an expression for the nth term of a sequence
- determine the monotonicity and boundedness of a sequence

2 helpful 'tidbits'

1) $$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = \lim_{u\to 0}(1+u)^{\frac{1}{u}} = e$$

2) Comparative growth rates:

$$\lim_{n\to\infty}\frac{faster}{slower} = \infty \qquad \lim_{n\to\infty}\frac{slower}{faster} = 0$$

logarithmic → polynomial → exponential → factorial

slower ⟹ faster

In your groups, evaluate the limit of each sequence below, and decide if the sequence converges or diverges.

1) $a_n = \frac{n + (-1)^n}{n}$

✓ $\lim_{n\to\infty} \frac{n + (-1)^n}{n} = \lim_{n\to\infty} \frac{1}{1} = 1 \Rightarrow \textit{converges to } 1$

2) $a_n = \frac{(-1)^n(n-1)}{n}$

✓ $\lim_{n\to\infty} \frac{(-1)^n(n-1)}{n} = \lim_{n\to\infty} (-1)^n(1) \Rightarrow \textit{diverges}$

3) $a_n = \left(1 + \frac{1}{n}\right)^n$

✓ $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \Rightarrow \textit{converges to } e$

4) $a_n = \frac{2(n!)}{(n-1)!} = \frac{2n(n-1)!}{(n-1)!}$

✓ $\lim_{n\to\infty} \frac{2(n!)}{(n-1)!} = \lim_{n\to\infty} \frac{2(n)(n-1)!}{(n-1)!} = \lim_{n\to\infty} 2n = \infty \Rightarrow \textit{diverges}$

5) $a_n = \frac{2^n}{(n+1)!}$ (comparative growth rates)

✓ $\lim_{n\to\infty} \frac{2^n}{(n+1)!} = 0 \Rightarrow \textit{converges to } 0$

think of this as: $\frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot \ldots}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot \ldots}$

Finding the nth term of a sequence:

- Look for an overall pattern
- If the terms are fractions, try looking for separate patterns with the numerators and denominators
- Most sequences consist of functions that are linear, polynomial, exponential, factorial, or combinations of these

ex) 3, 7, 11, 15, 19, . . .

+4 +4 +4

arithmetic

$a_n = 4n - 1$

3, 12, 48, 192, . . .

·4 ·4 ·4

geometric

$a_n = 3(4)^{n-1}$ or $\frac{3}{4}(4)^n$

factorials → multiplying by increasing #s

In your groups, write an expression for the nth term of each sequence below:

1) 3, 8, 13, 18, ... ✓ $a_n = 5n - 2$

2) 5, -15, 45, -135, ... ✓ $a_n = 5(-3)^{n-1}$ or $\frac{-5}{3}(-3)^n$

3) 4, 10, 28, 82, ... ✓ $a_n = 3^n + 1$

4) $\frac{2}{1}, \frac{3}{3}, \frac{4}{5}, \frac{5}{7}, \frac{6}{9}, \ldots$ ✓ $a_n = \frac{n+1}{(2n-1)}$

5) $-\frac{2}{1}, \frac{8}{2}, -\frac{26}{6}, \frac{80}{24}, -\frac{242}{120}, \ldots$ ✓ $a_n = (-1)^n \frac{(3^n - 1)}{n!}$

What do you think a monotonic sequence is?

A sequence where the terms are always nondecreasing or always nonincreasing.

(This means the terms are either always increasing or always decreasing, but it's okay if consecutive terms are equal to each other.)

What do you think is meant by a sequence that is bounded above or bounded below?

A sequence is bounded above if all terms are less than or equal to a specific number, M. M is called the upper bound of the sequence.

A sequence is bounded below if all terms are greater than or equal to a specific number N. N is called the lower bound of the sequence.

A sequence is bounded if it is bounded both above and below.

Theorem: Any sequence that is both monotonic and bounded will converge.

Note: this theorem is <u>not</u> biconditional

In your groups, use any method to determine the monotonicity and boundedness of the sequence below.

ex) $a_n = \dfrac{3n}{n+2}$

Monotonicity: 3 methods

1) list the first several terms and realize that the numerator grows by 3 while the denominator grows by 1 each term so every term will be greater than the previous term
2) set up an inequality 3n/(n+2) <?> 3(n+1)/((n+1)+2) and solve
3) find the first derivative and determine if it is always one sign

Boundedness

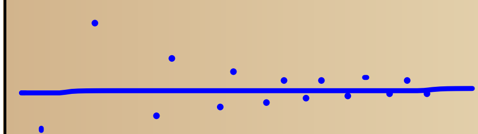
We know that the lower bound will be 1 (the first term) because the function is always increasing. If we check the limit as n approaches ∞ we get 3 so the upper bound is 3.

You try! Determine the monotonocity and boundedness of each sequence below:

1) $a_n = \frac{n}{e^{\frac{n}{2}}}$

not monotonic (inc to n = 2, then dec)
upper bound: 2/e
lower bound: 0
converges to 0

2) $a_n = \left(-\frac{2}{3}\right)^n$



not monotonic (bounces between positive and negative)
upper bound: 4/9
lower bound: -2/3
converges to 0

3) $a_n = \cos\left(\frac{n\pi}{2}\right)$

not monotonic (oscillates)
upper bound: 1
lower bound: -1
diverges

What have we learned?

- What does 'monotonic' mean?
- What does 'bounded' mean?
- How do we determine if a sequence converges?