# Warmup!!

Determine convergence/divergence of the following series:

1) $\sum_{n=1}^{\infty} ne^{-n^2}$ converges (I used the ratio test)

2) $\sum_{n=1}^{\infty} \frac{1}{3n+1}$ diverges (I used the limit comparison test)

3) $\sum_{n=1}^{\infty} \frac{n!}{10^n}$ even faster w/ $n^{th}$ term test

he ratio test)

4) $\sum_{n=1}^{\infty} \left(\frac{n+7}{2n+1}\right)^n$ converges (I used the root test)

① $\lim_{n\to\infty} \left| \frac{n+1}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n} \right|$

$= \lim_{n\to\infty} \left| \frac{n+1}{e^{2n+1} \cdot n} \right| = 0$

② $\lim_{n\to\infty} \left| \frac{\frac{1}{3n+1}}{\frac{1}{n}} \right| = \lim_{n\to\infty} \left| \frac{n}{3n+1} \right| = \frac{1}{3}$

HW

9-10

#17 $f(x)=\frac{1}{\sqrt{4+x^2}}=\frac{1}{2}\cdot\frac{1}{\sqrt{1+\left(\frac{x}{2}\right)^2}}$

$=\frac{1}{2}\left(1+\left(\frac{x}{2}\right)^2\right)^{-\frac{1}{2}}$ so $k=-\frac{1}{2}$

so

$$(1+x)^{-\frac{1}{2}}=1-\frac{1}{2}x+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^2}{2}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^3}{6}+\dots$$

$$=1-\frac{x}{2}+\frac{3x^2}{4(2!)}-\frac{(3\cdot5)x^3}{8(3!)}+\frac{(3\cdot5\cdot7)x^4}{16(4!)}\dots$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n x^n[1\cdot3\cdot5\cdot7\dots(2n-1)]}{2^n\cdot n!}+1$$

so

$$\left(1+\left(\frac{x}{2}\right)^2\right)^{-\frac{1}{2}}=\sum_{n=1}^{\infty}\frac{(-1)^n\left(\frac{x^2}{4}\right)^n[1\cdot3\cdot5\cdot7\dots(2n-1)]}{2^n\cdot n!}+1$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}[1\cdot3\cdot5\dots(2n-1)]}{4^n\cdot2^n\cdot n!}+1$$

so

$$\frac{1}{2}\left(1+\frac{x^2}{4}\right)^{-\frac{1}{2}}=\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}[1\cdot3\cdot5\dots(2n-1)]}{2(8)^n n!}+\frac{1}{2}$$

this could also be written as $2^{3n+1}$

(27) $f(x)=\frac{1}{2}(e^{x}-e^{-x})$ $f(0)=0$

$f'(x)=\frac{1}{2}(e^{x}+e^{-x})$ $f'(0)=1$

$f''(x)=\frac{1}{2}(e^{x}-e^{-x})$ $f''(0)=0$

$f'''(x)=\frac{1}{2}(e^{x}+e^{-x})$ $f'''(0)=1$

$\vdots$

$$P(x)=0+\frac{1x}{1}+0+\frac{1x^3}{3!}+0+\frac{1x^5}{5!}+\dots$$

$$=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\dots$$

$$=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$$

(35)

$$g(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) = \sin x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{ix} = \sum_{n=0}^{\infty} \frac{i^n x^n}{n!}$$

$$e^{-ix} = \sum_{n=0}^{\infty} \frac{(-i)^n x^n}{n!}$$

$$g(x) = \frac{1}{2i} \sum_{n=0}^{\infty} \left[\frac{i^n x^n}{n!} - \frac{(-i)^n x^n}{n!}\right]$$

$$= \frac{1}{2i} \sum_{n=0}^{\infty} \frac{\left[i^n - (-i)^n\right] x^n}{n!} = \sum_{n=0}^{\infty} \frac{\left[i^n - (-i)^n\right] x^n}{(2i)n!}$$

$$= \frac{(1-1)x^0}{2i} + \frac{(i - -i)x}{2i} + \frac{(i^2 - i^2)x^2}{2i(2!)} + \frac{(i^3 - -i^3)x^3}{2i(3!)} + \dots$$

$$= 0 + \frac{2ix}{2i} + 0 + \frac{2i^3x^3}{2i(3!)} + 0 + \frac{2i^5x^5}{2i(5!)} + \dots$$

$$= x + \frac{i^2x}{3!} + \frac{i^4x^5}{5!} + \frac{i^6x^7}{7!} + \frac{i^8x^9}{9!} + \dots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin x$$ ✓

# 9-10b: Operations with Taylor and Maclaurin Series

At the end of this lesson students will be able to:

- Use operations to create a Maclaurin series for products and quotients of functions
- Use power series to approximate an integral value with a specified accuracy

ex) Find the first four nonzero terms of the Maclaurin series for the g(x) = $e^x$cosx by multiplying the appropriate power series.

We know that:

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad f(x) = \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

However, the multiplication can't be done by simply multiplying the terms in the sigmas together. The terms need to be written out in polynomial form and then distributed properly. You only need 4 terms for your answer, but sometimes this requires using more than 4 terms in the multiplication.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right)$$

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$*1 \quad = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots$$

$$*-\frac{x^2}{2} = \quad -\frac{x^2}{2} - \frac{x^3}{2} - \frac{x^4}{4} - \frac{x^5}{12} - \cdots$$

$$*\frac{x^4}{24} = \quad \frac{x^4}{24} + \frac{x^5}{24} + \frac{x^6}{48} + \frac{x^7}{144}$$

$$*-\frac{x^6}{720} = \quad -\frac{x^6}{720} - \frac{x^7}{720} - \frac{x^8}{1440} - \frac{x^9}{4320}$$

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + \cdots$$

You try! Write out the first 4 nonzero terms of the Maclaurin series for the function $f(x) = \dfrac{e^x}{1+x}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$* 1 \quad = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$* -x \quad = \quad -x - x^2 - \frac{x^3}{2} - \frac{x^4}{6} - \frac{x^5}{24} - \cdots$$

$$* x^2 \quad = \quad x^2 + x^3 + \frac{x^4}{2} + \frac{x^5}{6} + \frac{x^6}{24} + \cdots$$

$$* -x^3 = \quad -x^3 - x^4 - \frac{x^5}{2} - \frac{x^6}{6} - \frac{x^7}{24} - \cdots$$

$$* x^4 \quad = \quad x^4 + x^5 + \frac{x^6}{2} + \frac{x^7}{6} + \cdots$$

$$= 1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8} - \cdots$$

# LAST TOPIC FOR CHAPTER 9!!

How to use a power series to approximate an integral with a specified error.

ex) Use a power series to approximate $\int_0^{\frac{1}{2}} \frac{\arctan x}{x}\,dx$ with an error less than 0.0001.

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\frac{\arctan x}{x} = 1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \frac{x^8}{9} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n+1}$$

$$\int_0^{\frac{1}{2}} \frac{\arctan x}{x}\,dx = x - \frac{x^3}{9} + \frac{x^5}{25} - \frac{x^7}{49} + \frac{x^9}{81} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)^2}\Bigg|_0^{\frac{1}{2}}$$

$$= \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3}{9} + \frac{\left(\frac{1}{2}\right)^5}{25} - \frac{\left(\frac{1}{2}\right)^7}{49} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)^{2n+1}}{(2n+1)^2} = Y1$$

Because this is an alternating series, we could simply look to see which term is < 0.0001 to determine how many terms to include in our approximation. The 5th term, $a_4$, < 0.0001 so our approximation would be:

$$\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3}{9} + \frac{\left(\frac{1}{2}\right)^5}{25} - \frac{\left(\frac{1}{2}\right)^7}{49} = 0.487202$$

To double-check, the actual value of the integral ≈ 0.4872223. Not bad!

LAST YOU TRY!!!

Use a power series to approximate the value of $\int_0^1 e^{-x^2}dx$ with an error less than 0.0001.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \cdots = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$\int_0^1 e^{-x^2}dx = x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)}}{n!\,(2n+1)}\Bigg|_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+1)}$$

Checking the table, it is the 8th term, $a_7$, that is < 0.0001, so we will need to sum the first 7 terms.

$$1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{1320} + \frac{1}{9360} = 0.74683603$$

The actual value ≈ 0.74682413 so we're good!

# What have we learned?

- Can I create a new Maclaurin series using multiplication and division?
- Can I approximate the value of an integral with specified accuracy?