Warmup!!

Let's try 2011 #6!

(53) $$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2^n n}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{1}{2}\right)^n$$

$$\ln x=(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\dots+\frac{(-1)^{n-1}(x-1)^n}{n}+\dots$$

$$\ln\left(\frac{3}{2}\right)=\frac{1}{2}-\frac{\left(\frac{1}{2}\right)^2}{2}+\frac{\left(\frac{1}{2}\right)^3}{3}-\frac{\left(\frac{1}{2}\right)^4}{4}+\dots$$

$$=\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1}{2^n n}$$

9-10a: Taylor and Maclaurin Series

Essential Learning Targets:

- Knows that if a power series has a positive radius of convergence, then the power series is the Taylor series of the function to which it converges over the open interval.
- Constructs Maclaurin series for other functions using Maclaurin series for sin(x), cos(x) and e^x.

Reminder: A Taylor Polynomial is an approximation for a function with a finite number of terms.

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

A Taylor series is an exact polynomial-type representation of a function with an infinite number of terms.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)(x-c)^n}{n!} = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

If you can find a power series to represent a specific function using any other method (geometric sum, ...), then this series will be equivalent to the Taylor series for the function.

ex) Write the first 4 terms and the general term for the Taylor Polynomial for $f(x) = e^{3x}$ centered at $c = 0$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

$$e^{3x} = 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \dots + \frac{(3x)^n}{n!} + \dots$$

You Try! Write the first four terms and the general term of the Taylor series for $f(x) = \sin x$, centered at $c = \pi/4$.

Note: to make the sign change after each pair of terms, use an exponent of $n(n-1)/2$ on your (-1) term

$f(x) = \sin x$ $\quad$ $f(\pi/4) = \sqrt{2}/2$

$f'(x) = \cos x$ $\quad$ $f'(\pi/4) = \sqrt{2}/2$

$f''(x) = -\sin x$ $\quad$ $f''(\pi/4) = -\sqrt{2}/2$

$f'''(x) = -\cos x$ $\quad$ $f'''(\pi/4) = -\sqrt{2}/2$

$$f(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}\left(x - \frac{\pi}{4}\right)}{2 \cdot 1!} - \frac{\sqrt{2}\left(x - \frac{\pi}{4}\right)^2}{2 \cdot 2!} - \frac{\sqrt{2}\left(x - \frac{\pi}{4}\right)^3}{2 \cdot 3!} + \cdots$$

$$+ \; (-1)^{\frac{n(n-1)}{2}} \frac{\sqrt{2}\left(x - \frac{\pi}{4}\right)^n}{2\,n!} + \ldots$$

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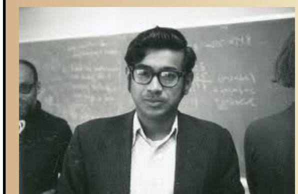
Brain Break: It's Mathematician Monday!!

Srinivasa Ramanujan!



Srinivasa was born in 1887 and grew up in a town not terribly far from Madras, India. He did well in school and after graduating from high school chose to study mathematics on his own, at first simply using a book he found in his high school. In 1902 he developed a method for solving a quartic polynomial equation. By 1904 he was focusing his research on infinite series, and independently discovered Bernoulli numbers without knowing they had already been discovered by Bernoulli. Because of his work, he was granted a scholarship to the Government College of Kumbakonam, but the scholarship was not renewed his 2nd year because he ignored all other college subjects besides math.

Srinivasa 'ran away' to a new town and focused independently on hypergeometric series, where he again independently discovered elliptic functions without knowing they had already been discovered. In 1906 he tried to enter the University of Madras but failed the entrance exams for all subjects besides math. So he decided to focus instead on studying continued fractions and divergent series. In 1909 his mother arranged for him to marry a 10-year old girl, but they didn't move in together until she was 12. In the meantime in 1911 he published a highly acclaimed paper on Bernoulli numbers in an Indian mathematical journal. This finally gave him some recognition and he became known as a mathematical genius. Because of this the founder of the Indian Mathematical Society arranged for him to get his first job (accounts clerk at the Madras Port Trust). This is what he wrote when Ramanujan walked in:

> *A short uncouth figure, stout, unshaven, not over clean, with one conspicuous feature-shining eyes- walked in with a frayed notebook under his arm. He was miserably poor. ... He opened his book and began to explain some of his discoveries. I saw quite at once that there was something out of the way; but my knowledge did not permit me to judge whether he talked sense or nonsense. ... I asked him what he wanted. He said he wanted a pittance to live on so that he might pursue his researches.*

In 1913, Ramanujan wrote to G H Hardy, who had written a book on orders of infinity, and Hardy was very impressed with his work. Through Hardy, Ramanujan was quickly admitted to the Univesity of Madras, and then quickly transferred to Trinity College in London. Unfortunately Ramanujan was seriously ill for many of his adult years and eventually moved back to India to try to recover but died very young in 1920.

While in London, Ramanujan became famous for his mathematical contributions. His only major setback was that because he had so little formal training he did not have a full grasp of how to write a proof and some of his theorems were completely wrong. But it was later shown that he had independently discovered Riemann series, elliptic integrals, hypergeometric series, and many of the results of Gauss, Kummer, and others. He created a series for approximating $1/\pi$ where each successive term adds 8 more correct digits to the approximation.

Let's remind ourselves of some of those elementary function series representations again.

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$f(x) = \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$f(x) = \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$f(x) = \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$f(x) = \ln(x+1) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

$$f(x) = \frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

$$f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

Write the first four nonzero terms of the Maclaurin series for $g(x) = 2\sin(x^3)$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin x^3 = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \cdots$$

$$2\sin x^3 = 2x^3 - \frac{2x^9}{3!} + \frac{2x^{15}}{5!} - \frac{2x^{21}}{7!} + \cdots$$

Warmup!!

1) Determine the convergence or divergence of

$$\sum_{n=8}^{\infty} \frac{(4n)^n}{9^n n!}$$

I spotted the n! so I used the ratio test to get:

$$\lim_{n\to\infty} \left| \frac{\frac{(4(n+1))^{n+1}}{9^{n+1}(n+1)!}}{\frac{(4n)^n}{9^n n!}} \right| = \lim_{n\to\infty} \left| \frac{4^{n+1}(n+1)^{n+1}}{9^{n+1}(n+1)!} \cdot \frac{9^n n!}{4^n n^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{4(n+1)^n(n+1)}{9(n+1)n^n} \right| = \lim_{n\to\infty} \left| \frac{4}{9} \left(\frac{n+1}{n} \right)^n \right| = \frac{4}{9} e > 1$$

What have we learned?

- Can I write a Taylor or Maclaurin polynomial as a series using sigma notation?
- Can I take a known series and create a new series by modifying the input or the function in the series algebraically?

LAST type of new series for the unit: the Binomial series!!!

A binomial series is the expansion of the function $f(x) = (1 + x)^k$

ex) Find the Maclaurin series (just polynomial form) for $f(x) = (1 + x)^k$

$f(x) = (1 + x)^k$	$f(0) = 1$
$f'(x) = k(1 + x)^{k-1}$	$f'(0) = k$
$f''(x) = k(k - 1)(1 + x)^{k-2}$	$f''(0) = k(k - 1)$
$f'''(x) = k(k - 1)(k - 2)(1 + x)^{k-3}$	$f'''(0) = k(k - 1)(k - 2)$

$$P_n(x) = 1 + kx + \frac{k(k-1)x^2}{2} + \frac{k(k-1)(k-2)x^3}{6} + \cdots$$

Use the binomial series we just found to write the Maclaurin series for $f(x) = \sqrt{1 + x^3}$

We know that if $f(x) = (1 + x)^k$, then

$$P_n(x) = 1 + \frac{kx}{1!} + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \cdots$$

So if we apply k = 1/2 to this series we get:

$$P_n(x) = 1 + \frac{1}{2}x + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{x^2}{2!} + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{x^3}{3!} + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{x^4}{4!}$$

$$P_n(x) = 1 + \frac{1}{2}x + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}[1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)]x^n}{2^n n!}$$

Question: Why did we have to write the first 2 terms in front of the sum instead of including them in the sum?

This takes care of the root, now we have to deal with the x^3. This is easier than you'd think. You may simply replace each x with x^3 and you're done!

$$P_n(x) = 1 + \frac{1}{2}x^3 + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}[1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)]x^{3n}}{2^n n!}$$