2015 BC #5

Consider the function $f(x) = \frac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by $f'(x) = \frac{k - 2x}{(x^2 - kx)^2}$.

a) Let $k = 3$, so that $f(x) = \frac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph at that point whose x-coordinate is 4.

b) Let $k = 4$, so that $f(x) = \frac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum or neither at $x = 2$. Justify your answer.

c) Find the value of k for which f has a critical point at $x = -5$

d) Let $k = 6$, so that $f(x) = \frac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f. Then find $\int f(x)dx$.

(49) $\lim_{x \to 1^+} (\ln x)^{x-1}$

$$y = \lim_{x \to 1^+} (\ln x)^{x-1}$$

$$\ln y = \lim_{x \to 1^+} (x-1) \ln(\ln x)$$

$$\ln y = \lim_{x \to 1^+} \frac{\ln(\ln x)}{\frac{1}{x-1}}$$

$$\ln y = \lim_{x \to 1^+} \frac{\frac{\frac{1}{x}}{\ln x}}{\frac{-1}{(x-1)^2}} = \lim_{x \to 1^+} \frac{-(x-1)^2}{x \ln x}$$

$$\ln y = \lim_{x \to 1^+} \frac{-2(x-1)}{\ln x + 1} = 0$$

$$\boxed{y = e^0 = 1}$$ ✓

(51) $$\lim_{x\to 2^+}\left(\frac{8}{x^2-4}-\frac{x}{x-2}\right)$$

$$=\lim_{x\to 2^+}\left(\frac{8-x(x+2)}{x^2-4}\right)$$

$$=\lim_{x\to 2^+}\left(\frac{-x^2-2x+8}{x^2-4}\right)$$

$$=\lim_{x\to 2^+}\left(\frac{-(x-4)\cancel{(x+2)}}{(x-2)\cancel{(x+2)}}\right)$$

$$=\lim_{x\to 2^+}\frac{-x+4}{x-2}=\frac{2}{0}\Rightarrow \text{VA}$$

$$\frac{+}{+}=\infty$$

(53)

$$\lim_{x\to 1^+}\left(\frac{3}{\ln x}-\frac{2}{x-1}\right)$$

$$=\lim_{x\to 1^+}\left(\frac{3x-3-2\ln x}{(x-1)\ln x}\right)$$

$$=\lim_{x\to 1^+}\left(\frac{3-\frac{2}{x}}{\ln x+\frac{x-1}{x}}\right)$$

$$=\lim_{x\to 1^+}\left(\frac{3x-2}{x\ln x+x-1}\right)=\frac{1}{0}\Rightarrow VA$$

$$\frac{+}{+} \qquad =\infty$$

(69) $$\lim_{x\to\infty} \frac{(\ln x)^n}{x^m}$$

$$= \lim_{x\to\infty} \frac{n(\ln x)^{n-1}\left(\frac{1}{x}\right)}{m x^{m-1}}$$

$$= \lim_{x\to\infty} \frac{n(\ln x)^{n-1}}{m \cdot x \cdot x^{m-1}} = \lim_{x\to\infty} \frac{n(\ln x)^{n-1}}{m x^m}$$

$$= \lim_{x\to\infty} \frac{n(n-1)(\ln x)^{n-2}\left(\frac{1}{x}\right)}{m^2 x^{m-1}}$$

$$= \lim_{x\to\infty} \frac{n(n-1)(\ln x)^{n-2}}{m^2 x^m} = \dots$$

← reduces in size

stays x^m

$= 0$

so $(\ln x)^n$ grows at a slower rate than x^m

(99) $\lim_{x \to 0} \frac{a - \cos bx}{x^2} = 2$

$a = \cos 0$

$a = 1$

$= \lim_{x \to 0} \frac{b \sin bx}{2x}$

$= \lim_{x \to 0} \frac{b^2 \cos bx}{2} = \frac{b^2}{2} = 2$

So $b^2 = 4$

$a = 1$, $b = \pm 2$

CORE THIS WEEK - TEST REVIEW

ROOM 323

Be there or be a regular quadrilateral

8.8a Improper Integrals

At the end of this lesson you will be able to:

- state why an integral is improper
- evaluate an improper integral using a variety of methods

(Note: there is a homework change for tonight)

An improper integral is one that has either:

- non-finite limits (one or both of the limits are ∞ or $-\infty$)
- a discontinuous integrand function

For non-finite limits, rewrite as follows:

case 1: $$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

case 2: $$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx$$

case 3: $$\int_{-\infty}^\infty f(x)dx = \lim_{a\to-\infty} \int_a^c f(x)dx + \lim_{b\to\infty} \int_c^b f(x)dx$$

where c is any real number

Note:

If the limit exists, then the integral converges.
If the limit does not exist, then the integral diverges.
For case 3, if either portion diverges, then the whole thing diverges

One More Note: just because the area under the curve is infinite, does not necessarily imply that the integral will diverge. You have to evaluate it to find out (or run a convergence test which we will not get into today).

Warmup:

Use your calculator to evaluate each of the following:

1) $\int_{1}^{100} \frac{1}{x} dx$ = ln 100 ≈ 4.605

✓

2) $\int_{1}^{1000} \frac{1}{x} dx$ = ln 1000 ≈ 6.907

3) $\int_{1}^{10000} \frac{1}{x} dx$ = ln 10000 ≈ 9.210

Based on your answers, determine

$\int_{1}^{\infty} \frac{1}{x} dx$ = ln ∞ ≈ ∞

Note: just because we're adding limits into the situation doesn't mean the integrals will be easy. For example, what method would we have to use to evaluate the integral below?

ex) $\int_0^{\infty} (x-1)e^{-x}\,dx = \lim_{b\to\infty} \int_0^b (x-1)e^{-x}\,dx$

polyn. integrate repeatedly

	u	dv
+	$x-1$	e^{-x}
−	1	$-e^{-x}$
+	0	e^{-x}

$$\lim_{b\to\infty} \left(-(x-1)e^{-x} - e^{-x}\right)\Big|_0^b$$

$$= \lim_{b\to\infty} \left[-(b-1)e^{-b} - e^{-b} - (1-1)\right]$$

$$= \lim_{b\to\infty} \left[e^{-b}(-b+1-1)\right]$$

remember, $\sqrt{x^2} = x$ if $x > 0$ and $\sqrt{x^2} = -x$ if $x < 0$

$$= \lim_{b\to\infty} e^{-b}(-b)$$

$$= \lim_{b\to\infty} \frac{-b}{e^b} = \lim_{b\to\infty} \frac{-1}{e^b} = 0$$

Try a simple one to start!

$$\int_1^\infty \frac{5}{x^3}\,dx$$

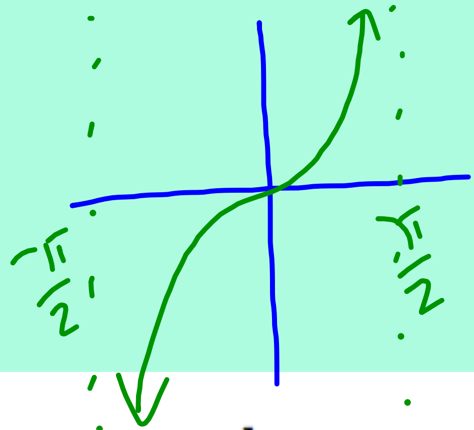
✓

$$= \lim_{b\to\infty}\int_1^b \frac{5}{x^3}\,dx = \lim_{b\to\infty}\left(-\frac{5}{2x^2}\Big|_1^b\right)$$

$$= \lim_{b\to\infty}\left(-\frac{5}{2b^2}+\frac{5}{2}\right) = \frac{5}{2}$$

Let's take it up a notch!

$$\int_0^{\infty} \frac{1}{x^2+1}\,dx$$



$$= \lim_{b\to\infty} \int_0^{b} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \arctan x \,\Big|_0^b$$

$$= \lim_{b\to\infty} (\arctan b - \arctan 0) = \lim_{b\to\infty} \arctan b = \frac{\pi}{2}$$

this is the angle (between $-\pi/2$ and $\pi/2$) where the tangent approaches infinity

Let's walk through a double infinity integral together. Remember that you can split it up at any real number you choose. (I like zero.)

$$ex)\ \int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$$

1) split the integral:

$$= \int_{-\infty}^{0} \frac{e^x}{1+e^{2x}}\,dx + \int_{0}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$$

2) rewrite with limits:

$$= \lim_{a\to-\infty} \int_{x=a}^{x=0} \frac{e^x}{1+e^{2x}}\,dx + \lim_{b\to\infty} \int_{0}^{b} \frac{e^x}{1+e^{2x}}\,dx$$

$$\int \frac{1}{1+u^2}\,du$$

$let\ u = e^x, so\ du = e^x dx$

3) evaluate the integral:

$$= \lim_{a\to-\infty} \int_{u=e^a}^{u=1} \frac{1}{1+u^2}\,du + \lim_{b\to\infty} \int_{1}^{e^b} \frac{1}{1+u^2}\,du$$

$$= \lim_{a\to-\infty} \arctan u \,\Big|_{e^a}^{1} + \lim_{b\to\infty} \arctan u \,\Big|_{1}^{e^b}$$

$$= \lim_{a\to-\infty} (\arctan 1 - \arctan e^a) + \lim_{b\to\infty} (\arctan e^b - \arctan 1)$$

4) evaluate the limits:

$$= \frac{\pi}{4} - \arctan e^{-\infty} + \arctan e^{\infty} - \frac{\pi}{4}$$

$$\approx -\arctan 0 + \arctan \infty$$

$$e^{-\infty} = \frac{1}{e^{\infty}} \approx 0$$

$$= 0 + \frac{\pi}{2} = \frac{\pi}{2}$$

Changing limits

$$\int_1^2 (x+5)^2\,dx$$

$u = x+5$
$du = dx$

$$= \int_6^7 u^2\,du$$

$$= \frac{1}{3}u^3 \Big|_6^7$$

$$= \frac{7^3}{3} - \frac{6^3}{3}$$

What have we learned?

- What is an improper integral and how can I recognize it?
- What wonderful things do I use to rewrite improper integrals in a form that can be integrated?