It's Warmup time!

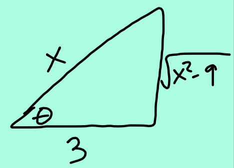
Write the following as a sum of 2 or more fractions with unlike denominators:

1) $\boxed{\frac{5}{6}} = \frac{3}{6} + \frac{2}{6} = \boxed{\frac{1}{2} + \frac{1}{3}}$

2) $\boxed{\frac{9}{20}} = \frac{4}{20} + \frac{5}{20} = \boxed{\frac{1}{5} + \frac{1}{4}}$

3) $\boxed{\frac{13}{15}} = \frac{5}{15} + \frac{8}{15} = \cancel{\boxed{\frac{1}{3} + \frac{8}{15}}}$ OR $\frac{3}{15} + \frac{10}{15} = \boxed{\frac{1}{5} + \frac{2}{3}}$

(51) $\displaystyle\int_4^6 \frac{x^2}{\sqrt{x^2-9}}\,dx$



$\sqrt{x^2-9} = 3\tan\theta$

$x = 3\sec\theta$

$x^2 = 9\sec^2\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$

$$= \int \frac{(9\sec^2\theta)(3\sec\theta\tan\theta)}{3\tan\theta}\,d\theta$$

$$= \int 9\sec^3\theta\,d\theta$$

$u = 9\sec\theta \qquad du = 9\sec\theta\tan\theta\,d\theta$

$v = \tan\theta \qquad dv = \sec^2\theta\,d\theta$

$$\int 9\sec^3\theta\,d\theta = 9\sec\theta\tan\theta - \int 9\sec\theta\tan^2\theta\,d\theta$$

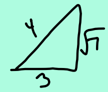
$$\int 9\sec^3\theta\,d\theta = 9\sec\theta\tan\theta - 9\int \sec\theta(\sec^2\theta - 1)\,d\theta$$

$$\int 9\sec^3\theta\,d\theta = 9\sec\theta\tan\theta - 9\int \sec^3\theta\,d\theta + \int 9\sec\theta\,d\theta$$

$$18\int \sec^3\theta\,d\theta = 9\sec\theta\tan\theta + 9\ln|\sec\theta + \tan\theta|$$

$$9\int_{\operatorname{arcsec}\frac{4}{3}}^{\frac{\pi}{3}} \sec^3\theta\,d\theta = \frac{9}{2}\sec\theta\tan\theta + \frac{9}{2}\ln|\sec\theta + \tan\theta|\Bigg|_{\operatorname{arcsec}\frac{4}{3}}^{\frac{\pi}{3}}$$

$$= \frac{9}{2}\sec\frac{\pi}{3}\tan\frac{\pi}{3} + \frac{9}{2}\ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right|$$



$$- \left[\frac{9}{2}\sec\left(\operatorname{arcsec}\frac{4}{3}\right)\tan\left(\operatorname{arcsec}\frac{4}{3}\right) + \frac{9}{2}\ln\left|\sec\left(\operatorname{arcsec}\frac{4}{3}\right) + \tan\left(\operatorname{arcsec}\frac{4}{3}\right)\right|\right]$$

$$= \frac{9}{2}(2)(\sqrt{3}) + \frac{9}{2}\ln|2+\sqrt{3}| - \frac{9}{2}\left(\frac{4}{3}\right)\left(\frac{\sqrt{7}}{3}\right) - \frac{9}{2}\ln\left|\frac{4}{3} + \frac{\sqrt{7}}{3}\right|$$

$$= 9\sqrt{3} + \frac{9}{2}\ln|2+\sqrt{3}| - 2\sqrt{7} - \frac{9}{2}\ln\left|\frac{4+\sqrt{7}}{3}\right|$$

$$\approx 12.644$$

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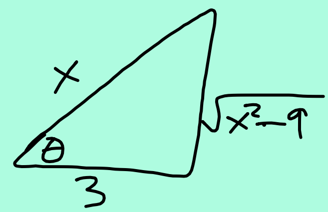
$$x\frac{dy}{dx}=\sqrt{x^2-9},\quad x\geq 3,\quad y(3)=1$$

$$\int dy=\int\frac{\sqrt{x^2-9}}{x}\,dx$$



$$\sqrt{x^2-9}=3\tan\theta$$
$$x=3\sec\theta$$
$$dx=3\sec\theta\tan\theta\,d\theta$$

$$y=\int\frac{(3\tan\theta)(3\sec\theta\tan\theta)}{3\sec\theta}\,d\theta$$

$$=\int 3\tan^2\theta\,d\theta$$

$$=\int 3(\sec^2\theta-1)\,d\theta$$

$$=\int(3\sec^2\theta-3)\,d\theta$$

$$=3\tan\theta-3\theta+C$$

$$y=\sqrt{x^2-9}-3\operatorname{arcsec}\frac{x}{3}+C$$

$$1=0-3\operatorname{arcsec}1+C$$

$$1=0-0+C\quad C=1$$

$$y=\sqrt{x^2-9}-3\operatorname{arcsec}\frac{x}{3}+1$$

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$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

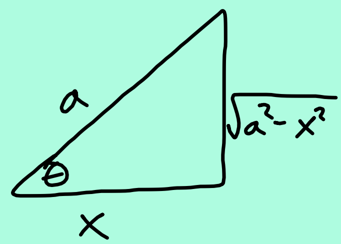
$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

$$\text{Area} = 2\int_{-a}^{a} \frac{b}{a}\sqrt{a^2 - x^2}\,dx = 4\int_{0}^{a} \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$\sqrt{a^2 - x^2} = a\sin\theta$$

$$x = a\cos\theta$$

$$dx = -a\sin\theta\,d\theta$$



$$A = 4\int_{\frac{\pi}{2}}^{0} \frac{b}{a}(a\sin\theta)(-a\sin\theta)\,d\theta$$

$$= 4\int_{0}^{\frac{\pi}{2}} ab\sin^2\theta\,d\theta$$

$$= 4\int_{0}^{\frac{\pi}{2}} \frac{ab}{2}(1 - \cos 2\theta)\,d\theta$$

$$= 2ab\theta - ab\sin 2\theta \Big|_{0}^{\frac{\pi}{2}}$$

$$= ab\pi - 0 - 0 - 0 = ab\pi$$

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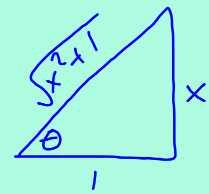
$y' = \frac{1}{x}$

$AL = \int_1^5 \sqrt{1 + \frac{1}{x^2}}\, dx$



$= \int_1^5 \sqrt{\frac{x^2+1}{x^2}}\, dx = \int_1^5 \frac{\sqrt{x^2+1}}{x}\, dx$

$\sqrt{x^2+1} = \sec\theta$

$x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$= \int_{\frac{\pi}{4}}^{\arctan 5} \frac{(\sec\theta)(\sec^2\theta)}{\tan\theta}\, d\theta$

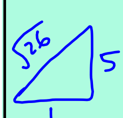
$= \int_{\frac{\pi}{4}}^{\arctan 5} \frac{\sec\theta(\tan^2\theta + 1)}{\tan\theta}\, d\theta$

$= \int_{\frac{\pi}{4}}^{\arctan 5} \sec\theta\tan\theta + \csc\theta\, d\theta$

$= \sec\theta - \ln|\csc\theta + \cot\theta| \Big|_{\frac{\pi}{4}}^{\arctan 5}$

$= \sec(\arctan 5) - \ln|\csc(\arctan 5) + \cot(\arctan 5)|$

$- \left[\sec\frac{\pi}{4} - \ln\left|\csc\frac{\pi}{4} + \cot\frac{\pi}{4}\right|\right]$



$= \sqrt{26} - \ln\left|\frac{\sqrt{26}}{5} + \frac{1}{5}\right| - \sqrt{2} + \ln\left|\sqrt{2} + 1\right|$

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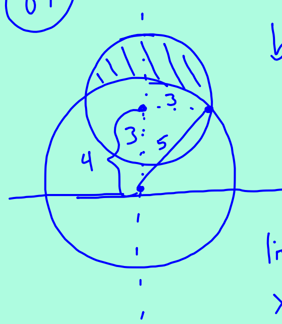


big circle:

$$x^2+y^2=25$$

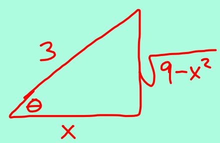
$$y=\pm\sqrt{25-x^2}$$

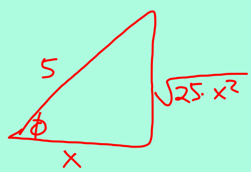
little circle:

$$x^2+(y-4)^2=9$$

$$y=\pm\sqrt{9-x^2}+4$$

$$A=2\int_0^3\left[4+\sqrt{9-x^2}-\sqrt{25-x^2}\right]dx$$





$\sqrt{9-x^2}=3\sin\theta$

$x=3\cos\theta$

$dx=-3\sin\theta\,d\theta$

$\int\left(4+\sqrt{9-x^2}\right)dx$

$=\int(4+3\sin\theta)(-3\sin\theta)\,d\theta$

$=\int(-12\sin\theta-9\sin^2\theta)\,d\theta$

$=\int\left[-12\sin\theta-\frac{9}{2}(1-\cos 2\theta)\right]d\theta$

$=12\cos\theta-\frac{9}{2}\theta+\frac{9}{4}\sin 2\theta$

$=12\left(\frac{x}{3}\right)-\frac{9}{2}\arccos\left(\frac{x}{3}\right)$
$\quad+\frac{9}{4}(2)\sin\theta\cos\theta$

$=4x-\frac{9}{2}\arccos\frac{x}{3}+\frac{9}{2}\left(\frac{\sqrt{9-x^2}}{3}\right)\left(\frac{x}{3}\right)$

$\sqrt{25-x^2}=5\sin\phi$

$x=5\cos\phi$

$dx=-5\sin\phi\,d\phi$

$\int\sqrt{25-x^2}\,dx$

$=\int 5\sin\phi(-5\sin\phi)\,d\phi$

$=\int -25\sin^2\phi\,d\phi$

$=\int-\frac{25}{2}(1-\cos 2\phi)\,d\phi$

$=-\frac{25}{2}\phi+\frac{25}{4}\sin 2\phi$

$=-\frac{25}{2}\left(\arccos\frac{x}{5}\right)+\frac{25}{4}(2)\sin\phi\cos\phi$

$=-\frac{25}{2}\arccos\frac{x}{5}+\frac{25}{2}\left(\frac{\sqrt{25-x^2}}{5}\right)\left(\frac{x}{5}\right)$

$$=2\left[4x-\frac{9}{2}\arccos\frac{x}{3}+\frac{x\sqrt{9-x^2}}{2}+\frac{25}{2}\arccos\frac{x}{5}-\frac{x\sqrt{25-x^2}}{2}\right|_0^3$$

$$=2\left[12-\frac{9}{2}(0)+0+\frac{25}{2}\arccos\frac{3}{5}-6-\left(0-\frac{9}{2}\left(\frac{\pi}{2}\right)+0+\frac{25}{2}\left(\frac{\pi}{2}\right)-0\right)\right]$$

$$=24+25\arccos\frac{3}{5}-12+\frac{9\pi}{2}-\frac{25\pi}{2}$$

$$=12+25\arccos\frac{3}{5}-8\pi\approx 10.0496$$

8.5a Partial Fractions!

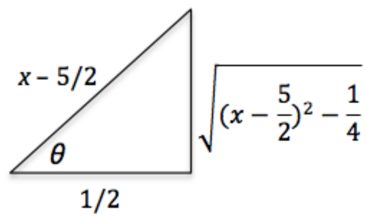
At the end of this lesson you will be able to:

- rewrite a rational function as a sum of partial fractions
- use partial fractions to integrate rational functions

Just for fun, let's see what it would take using current methods to evaluate the following integral and <u>simplify completely</u>

$$\int \frac{1}{x^2 - 5x + 6}\,dx$$

$$= \int \frac{1}{\left(\sqrt{\left(x - \frac{5}{2}\right)^2 - \frac{1}{4}}\right)^2}\,dx$$



$$\sqrt{\left(x - \frac{5}{2}\right)^2 - \frac{1}{4}} = \frac{1}{2}\tan\theta$$

$$x - \frac{5}{2} = \frac{1}{2}\sec\theta \quad so\ x = \frac{1}{2}\sec\theta + \frac{5}{2}$$

$$dx = \frac{1}{2}\sec\theta\tan\theta\, d\theta$$

$$= \int \frac{\frac{1}{2}\sec\theta\tan\theta}{\frac{1}{4}\tan^2\theta}\,d\theta = 2\int \frac{\sec\theta\tan\theta}{\tan^2\theta}\,d\theta$$

$$= 2\int \frac{\sec\theta}{\tan\theta}\,d\theta = 2\int \frac{1}{\sin\theta}\,d\theta = 2\int \csc\theta\, d\theta$$

$$= -2\ln|\csc\theta + \cot\theta| + C$$

$$= -2\ln\left|\frac{x - \frac{5}{2}}{\sqrt{\left(x - \frac{5}{2}\right)^2 - \frac{1}{4}}} + \frac{\frac{1}{2}}{\sqrt{\left(x - \frac{5}{2}\right)^2 - \frac{1}{4}}}\right| + C$$

$$= -2\ln\left|\frac{x - 2}{\sqrt{(x^2 - 5x + 6)}}\right| + C = -2\ln\left|\frac{x - 2}{\sqrt{(x-2)(x-3)}}\right| + C$$

$$= -2\left(\ln|x - 2| - \frac{1}{2}(\ln|x - 2| + \ln|x - 3|)\right) + C$$

$$= -2\left(\frac{1}{2}\ln|x - 2| - \frac{1}{2}\ln|x - 3|\right) + C$$

$$= \ln|x - 3| - \ln|x - 2| + C$$

When can we use partial fractions?

Whenever you have a denominator that can be factored.

What is missing from this otherwise Las Vegas-type, theatrical review? ... Steps.

Steps for decomposing $p(x)/q(x)$ into partial fractions:

1) If the degree of $p(x) \geq$ the degree of $q(x)$, divide

2) Factor the denominator completely

3) Break down the linear factors: there must be as many fractions as the degree of each factor and the numerators should all be constants

4) Break down the quadratic factors: there must be as many fractions as the degree of each factor and the numerators should all be linear

ex) Write the partial fraction decomposition for

$$\frac{1}{x^2 - 5x + 6}$$

1) Check the degrees   top < bottom ✓

2) Factor the bottom $\frac{1}{(x-2)(x-3)}$

3) Split up the factors $\frac{A}{x-2} + \frac{B}{x-3}$

So $(x-2)(x-3)\frac{1}{\cancel{x^2 - 5x + 6}} = \frac{(\cancel{x-2})(x-3)A}{\cancel{x-2}} + \frac{B}{\cancel{x-3}}(x-2)(\cancel{x-3})$

multiply through by LCD to get:

$$1 = A(x-3) + B(x-2)$$

To solve for A, plug in whatever x-value sets the 'B' term to 0 (x = 2): so -A = 1 and A = -1

Do the same for B to get that B = 1

So $\frac{1}{x^2 - 5x + 6} = \frac{-1}{x-2} + \frac{1}{x-3}$ or $\frac{1}{x-3} - \frac{1}{x-2}$

What's the point? Well, let's take another look at our 'just for fun' problem.

Evaluate the following integral and simplify completely

Now $\displaystyle\int \frac{1}{x^2 - 5x + 6}\,dx = \int \left(\frac{1}{x-3} - \frac{1}{x-2}\right) dx$

$= \ln|x-3| - \ln|x-2| + C$

What do we do with repeated linear factors?

ex) Use partial fractions to evaluate the following:

$$\int \frac{3x+6}{x^3-6x^2+9x}\,dx = \int \boxed{\frac{3x+6}{x(x-3)^2}} \cancel{dx =} \int \boxed{\left(\frac{A}{x}+\frac{B}{(x-3)}+\frac{C}{(x-3)^2}\right)} dx$$

For each linear factor raised to a power, it needs to be written starting with a power of 1 up until the final power (so the number of partial fractions for that factor will equal the power)

You take it from here!

(Find A, B and C and integrate)

$$= \int \frac{3x+6}{x(x-3)^2}\,dx = \int \left(\frac{A}{x}+\frac{B}{(x-3)}+\frac{C}{(x-3)^2}\right) dx$$

$$3x+6 = A(x-3)^2 + Bx(x-3) + Cx$$

$$let\ x = 3\ to\ get\ C = 5$$

$$let\ x = 0\ to\ get\ A = \frac{2}{3}$$

$$plug\ in\ A\ and\ C\ and\ let\ x = anything\ else$$

$$(I\ used\ x = 1) to\ get\ B = -\frac{2}{3}$$

$$so\ \int \frac{3x+6}{x^3-6x^2+9x}\,dx = \int \left(\frac{2}{3x}-\frac{2}{3(x-3)}+\frac{5}{(x-3)^2}\right) dx$$

$$= \frac{2}{3}\ln|x| - \frac{2}{3}\ln|x-3| - \frac{5}{x-3} + C$$

What if one of the factors is quadratic?

ex) Use partial fractions to evaluate:

use constants with linear factors (even repeated linear)

use linear with quadratic factors

$$\int \frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)}\,dx = \int \left(\frac{A}{x} + \frac{B}{x-1} + \frac{Cx + D}{x^2 + 4}\right) dx$$

You take it from here!

$$2x^3 - 4x - 8 = \overset{2}{\cancel{A}}(x-1)(x^2+4) + \overset{-2}{\cancel{B}}x(x^2+4) + (Cx+D)x(x-1)$$

$let\ x = 0\ to\ get\ A = 2$

$let\ x = 1\ to\ get\ B = -2$

plug in 2 other values for x (I used -1 and 2) and solve the system to get C = 2, D = 4

$$\int \left(\frac{2}{x} - \frac{2}{x-1} + \frac{2x}{x^2+4} + \frac{4}{x^2+4}\right) dx$$

$$= 2\ln|x| - 2\ln|x-1| + \ln(x^2+4) + 2\arctan\frac{x}{2} + C$$

Try a few start to finish!

(Hmmm, make sure you check the degrees first!)

$$\int \frac{4x^3 - 3x + 5}{x^2 - 2x} dx$$

$$
\begin{array}{r|l}
 & 4x + 8 \\
x^2 - 2x & 4x^3 - 3x + 5 \\
 & 4x^3 - 8x^2 \\
\hline
 & 8x^2 - 3x + 5 \\
 & 8x^2 - 16x \\
\hline
 & 13x + 5
\end{array}
$$

$$= \int \left(4x + 8 + \frac{13x + 5}{x^2 - 2x}\right) dx$$

$$\frac{13x + 5}{x^2 - 2x} = \frac{A}{x} + \frac{B}{x - 2}$$

$$13x + 5 = A(x - 2) + Bx$$

$$if\ x = 0, then\ A = -\frac{5}{2}$$

$$if\ x = 2, then\ B = \frac{31}{2}$$

$$so \int \left(4x + 8 - \frac{5}{2x} + \frac{31}{2(x - 2)}\right) dx$$

$$= 2x^2 + 8x - \frac{5}{2}\ln|x| + \frac{31}{2}\ln|x - 2| + C$$

Get as far as you can on this one and I'll help you the rest of the way!

$$\int \frac{x^2 - 29x + 5}{(x-4)(x^2+3)}dx$$

$$\frac{x^2 - 29x + 5}{(x-4)(x^2+3)} = \frac{A}{x-4} + \frac{Bx + C}{x^2+3}$$

$$x^2 - 29x + 5 = A(x^2+3) + (Bx + C)(x-4)$$

$$let\ x = 4, so\ A = -5$$

but what now? Plug in the A-value and multiply out the right side to get:

$$x^2 - 29x + 5 = -5x^2 - 15 + Bx^2 - 4Bx + Cx - 4C$$

then group the terms variable:

$$x^2 - 29x + 5 = (-5 + B)x^2 + (-4B + C)x - 4C - 15$$

$$so - 5 + B = 1\ and\ B = 6, -4C - 15 = 5\ so\ C = -5$$

$$so \int \left(\frac{-5}{x-4} + \frac{6x-5}{x^2+3}\right)dx = \int \left(\frac{-5}{x-4} + \frac{6x}{x^2+3} - \frac{5}{x^2+3}\right)dx$$

$$= -5\ln|x-4| + 3\ln(x^2+3) - \frac{5}{\sqrt{3}}\arctan\frac{x}{\sqrt{3}} + C$$

What have we learned?

Can I:

- rewrite a rational function as a sum of partial fractions?
- use partial fractions to integrate rational functions?

February 08, 2016