WARMUP! Let's review our identities!

The following identities will be needed in order to be successful with this section. See how many you can fill in!

1) $\sin^2x = 1 - \cos^2x$

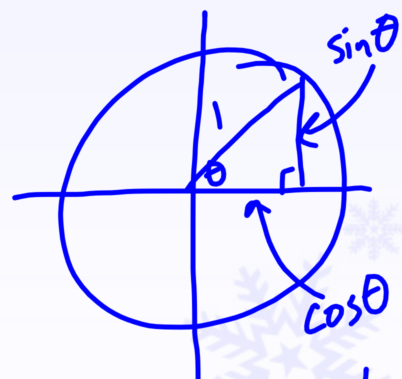
2) $\cos^2x = 1 - \sin^2x$

3) $\sin2x = 2\sin x\cos x$

4) $\cos2x = \cos^2x - \sin^2x$

$\cos2x = 2\cos^2x - 1$

$\cos2x = 1 - 2\sin^2x$



$$\frac{\sin^2x}{\cos^2x}+\frac{\cos^2x}{\cos^2x}=\frac{1}{\cos^2x}$$

$$\tan^2x + 1 = \sec^2x$$

Based on the cos2x identities,

5) $\sin^2x = 1/2\,(1 - \cos2x)$

6) $\cos^2x = 1/2\,(1 + \cos2x)$

power reduction identities

5) $\tan^2x = \sec^2x - 1$

6) $\sec^2x = \tan^2x + 1$

51 $\displaystyle\int_0^{\frac{1}{2}} \arccos x \, dx$

$u = \arccos x \qquad du = \dfrac{-1}{\sqrt{1-x^2}}\,dx$

$v = x \qquad dv = dx$

$= x\arccos x + \displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$

$w = 1 - x^2$

$dw = 2x\,dx$

$\frac{1}{2}dw = x\,dx$

$= x\arccos x + \displaystyle\int \frac{1}{2} w^{-\frac{1}{2}}\,dw$

$= x\arccos x + \frac{1}{2}\cdot 2 w^{\frac{1}{2}}$

$= x\arccos x + \sqrt{1-x^2}\,\Big|_0^{\frac{1}{2}}$

$= \frac{1}{2}\left(\frac{\pi}{3}\right) + \sqrt{\frac{3}{4}} - (0 + 1)$

$= \frac{\pi}{6} + \frac{\sqrt{3}}{2} - 1$

53 $\int_0^1 e^x \sin x \, dx$

$u = \sin x \qquad du = \cos x \, dx$

$v = e^x \qquad dv = e^x \, dx$

$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$

$u = \cos x \qquad du = -\sin x \, dx$

$v = e^x \qquad dv = e^x \, dx$

$\int e^x \sin x \, dx = e^x \sin x - e^x \cos x + \int -e^x \sin x \, dx$

$2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x$

$\int_0^1 e^x \sin x \, dx = \frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x \Big|_0^1$

$= \frac{1}{2} e \sin 1 - \frac{1}{2} e \cos 1 - \left(-\frac{1}{2}\right)$

57 $\displaystyle\int_2^4 x \operatorname{arcsec} x \, dx$

dropping abs value b/c $x>0$ on $[2,4]$

$u = \operatorname{arcsec} x \qquad du = \dfrac{1}{x\sqrt{x^2-1}}\,dx$

$v = \frac{1}{2}x^2 \qquad dv = x\,dx$

$$\int x \operatorname{arcsec} x\,dx = \frac{1}{2}x^2 \operatorname{arcsec} x - \int \frac{x^2}{2x\sqrt{x^2-1}}\,dx$$

$$= \frac{1}{2}x^2 \operatorname{arcsec} x - \int \frac{x}{2\sqrt{x^2-1}}\,dx$$

$w = x^2 - 1$

$dw = 2x\,dx$

$\frac{1}{4}dw = \frac{1}{2}x\,dx$

$$= \frac{1}{2}x^2 \operatorname{arcsec} x - \int \frac{1}{4} w^{-\frac{1}{2}}\,dw$$

$$= \frac{1}{2}x^2 \operatorname{arcsec} x - \frac{1}{2}w^{\frac{1}{2}} + C$$

$$\int_2^4 x \operatorname{arcsec} x\,dx = \frac{1}{2}x^2 \operatorname{arcsec} x - \frac{1}{2}\sqrt{x^2-1}\,\Big|_2^4$$

$$= 8\operatorname{arcsec} 4 - \frac{1}{2}\sqrt{15} - 2\underbrace{\operatorname{arcsec} 2}_{\frac{\pi}{3}} + \frac{1}{2}\sqrt{3}$$

$$= 8\operatorname{arcsec} 4 - \frac{1}{2}\sqrt{15} - \frac{2\pi}{3} + \frac{1}{2}\sqrt{3}$$

8.3a Trig Integrals!!

Essential Learning Target:

- Techniques for finding antiderivatives include algebraic manipulation such as long division, completing the square, substitution of variables and integration by parts

Diving in!

If one of the exponents is odd, that's the function you want to convert. Always save one extra for your u-sub.

(If they're both odd, then it doesn't matter which one you pick.) (If they're both even, we'll discuss that later.)

$\int \sin^5 x \cos^4 x dx =$

1) pull one sinx out to use later on

$$\int \sin x (\sin^4 x \cos^4 x) dx$$

2) rewrite the $\sin^4 x$ as $(1 - \cos^2 x)^2$

$$\int \sin x (1 - \cos^2 x)^2 (\cos^4 x) dx$$

3) square the binomial

$$\int \sin x [1 - 2\cos^2 x + \cos^4 x](\cos^4 x) dx$$

4) distribute the cosines

$$\int \underbrace{\sin x}_{du} [\cos^4 x - 2\cos^6 x + \cos^8 x] dx$$

5) let u = cosx and work from there

$u = \cos x$

$du = -\sin x dx$

$$\int -[u^4 - 2u^6 + u^8] du$$

$$= -\frac{1}{5}u^5 + \frac{2}{7}u^7 - \frac{1}{9}u^9 + C$$

$$= -\frac{1}{5}\cos^5 x + \frac{2}{7}\cos^7 x - \frac{1}{9}\cos^9 x + C$$

ex) $\int \sin^3 x \cos^3 x\,dx$

$$= \int \sin x \left(\sin^2 x \cos^3 x\right) dx$$

$$= \int \sin x \left[(1-\cos^2 x)\cos^3 x\right] dx$$

$$= \int \sin x \left(\cos^3 x - \cos^5 x\right) dx$$

$u = \cos x$

$du = -\sin x\,dx$

$$= -\int (u^3 - u^5)\,du$$

$$= -\frac{1}{4}u^4 + \frac{1}{6}u^6 + C$$

$$= -\frac{1}{4}\cos^4 x + \frac{1}{6}\cos^6 x + C$$

You try!

$\int \sin^2(4x)\cos^3(4x)dx$

$$= \int \cos(4x)\sin^2(4x)\cos^2(4x)\,dx$$

$$= \int \cos(4x)\sin^2(4x)\left[1-\sin^2(4x)\right]dx$$

$$= \int \cos(4x)\left[\sin^2(4x)-\sin^4(4x)\right]dx$$

$u = \sin(4x)$

$du = 4\cos(4x)\,dx$

$\frac{1}{4}du = \cos(4x)\,dx$

$$= \frac{1}{4}\int \left(u^2 - u^4\right)du$$

$$= \frac{1}{12}u^3 - \frac{1}{20}u^5 + C$$

$$= \frac{1}{12}\sin^3(4x) - \frac{1}{20}\sin^5(4x) + C$$

Try again!

$\int \sin^3 x dx$

$$= \int \sin x \cdot \sin^2 x \, dx$$

$$= \int \sin x (1 - \cos^2 x) \, dx$$

$u = \cos x$

$du = -\sin x \, dx$

$$= \int -(1 - u^2) \, du$$

$$= -u + \frac{1}{3}u^3 + C$$

$$= -\cos x + \frac{1}{3}\cos^3 x + C$$

What if both exponents are even?

Then you need to use the other identities to convert a function to an odd power of cosine.

ex) $\int \sin^2 x \cos^4 x\,dx$

1) rewrite $\sin^2 x$ as 1/2 (1 - cos2x)

$$\int \frac{1}{2}(1-\cos 2x)\cos^4 x\,dx$$

2) rewrite $\cos^4 x$ as $(1/2\,(1 + \cos 2x))^2$

$$\int \frac{1}{2}(1-\cos 2x)\left[\frac{1}{2}(1+\cos 2x)\right]^2 dx$$

3) multiply out the squared binomial

$$\int \frac{1}{2}(1-\cos(2x))\left[\frac{1}{4}\left(1 + 2\cos 2x + \cos^2 2x\right)\right]dx$$

$$=\int \frac{1}{8}\left[1 + \underline{2\cos 2x} + \underline{\cos^2 2x} - \underline{\cos 2x} - \underline{2\cos^2 2x} - \cos^3 2x\right]dx$$

4) distribute everything and simplify

$$=\int \frac{1}{8}\left[1 + \cos 2x - \cos^2 2x - \cos^3 2x\right]dx$$

5) reduce the squared term and integrate as much as possible

$$=\int \frac{1}{8}\left[1 + \cos 2x - \frac{1}{2}(1+\cos 4x) - \cos^3 2x\right]dx$$

$$=\int \frac{1}{8}\left[1 + \cos 2x - \frac{1}{2} - \frac{1}{2}\cos 4x - \cos^3 2x\right]dx$$

$$=\int \left[\frac{1}{16} + \frac{1}{8}\cos 2x - \frac{1}{16}\cos 4x - \frac{1}{8}\cos^3 2x\right]dx$$

$$=\frac{1}{16}x + \frac{1}{16}\sin 2x - \frac{1}{64}\sin 4x - \int \frac{1}{8}\cos^3 2x\,dx$$

6) integrate the cubed term as you would any odd-powered trig function by pulling out one term, converting the other function and using u-sub

$$=\frac{1}{16}x + \frac{1}{16}\sin 2x - \frac{1}{64}\sin 4x - \frac{1}{8}\int \cos 2x \cdot (1-\sin^2 2x)\,dx$$

$u = \sin 2x$

$du = 2\cos 2x\,dx$

$$=\frac{1}{16}x + \frac{1}{16}\sin 2x - \frac{1}{64}\sin 4x - \frac{1}{8}\int \frac{1}{2}(1-u^2)\,du$$

$$=\frac{1}{16}x + \frac{1}{16}\sin 2x - \frac{1}{64}\sin 4x - \frac{1}{16}\left[u - \frac{1}{3}u^3\right] + C$$

$$=\frac{1}{16}x + \frac{1}{16}\sin 2x - \frac{1}{64}\sin 4x - \frac{1}{16}\sin 2x + \frac{1}{48}\sin^3 2x + C$$

$$=\frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C$$

ex) $\int \sin^4(5x)dx$

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

1) rewrite $\sin^4(5x)$ as $(1/2\ (1 - \cos(10x)))^2$

$$\int \left[\left(\frac{1}{2}(1-\cos(10x))\right)^2\right]dx$$

2) multiply out the binomial

$$= \int \frac{1}{4}\left(1 - 2\cos(10x) + \cos^2(10x)\right)dx$$

3) rewrite the $\cos^2(10x)$ as $1/2\ (1 + \cos(20x))$

$$= \int \left[\frac{1}{4}\left(1 - 2\cos(10x) + \frac{1}{2}(1+\cos(20x))\right)\right]dx$$

4) integrate!

$$= \int \left[\frac{1}{4} - \frac{1}{2}\cos(10x) + \frac{1}{8} + \frac{1}{8}\cos(20x)\right]dx$$

$$= \frac{3}{8}x - \frac{1}{20}\sin(10x) + \frac{1}{160}\sin(20x) + C$$

That's great and all, but what about tangent?

Because $\tan^2 x = \sec^2 x - 1$ and the derivative of tanx is $\sec^2 x$, it's useful to take advantage of the link

If the power of secx is even, save a $\sec^2 x$ for the du and convert the rest to tangents

ex) $\int \sec^4(3x)\tan^3(3x)dx$

1) split up the $\sec^4(3x)$ and convert one of the squares to tangents

$$\int \sec^2(3x)\cdot\sec^2(3x)\cdot\tan^3(3x)\,dx$$

$$= \int \sec^2(3x)\left[\tan^2(3x)+1\right]\cdot\tan^3(3x)\,dx$$

2) distribute ~~everything~~ the tangents

$$= \int \sec^2(3x)\left[\tan^5(3x)+\tan^3(3x)\right]dx$$

3) u-sub and integrate

$u = \tan(3x)$

$du = 3\sec^2(3x)\,dx$

$$\int \frac{1}{3}\left[u^5+u^3\right]du$$

$$= \frac{1}{18}u^6 + \frac{1}{12}u^4 + C$$

$$= \frac{1}{18}\tan^6(3x) + \frac{1}{12}\tan^4(3x) + C$$

$$\int \sec^4 x \tan^3 x \, dx$$

$$= \int \sec x \tan x \cdot \sec^3 x \tan^2 x \, dx$$

$$= \int \sec x \tan x \cdot \sec^3 x \cdot (\sec^2 x - 1) \, dx$$

$$= \int \sec x \tan x \left[\sec^5 x - \sec^3 x \right] dx$$

$u = \sec x$

$du = \sec x \tan x \, dx$

$$= \int (u^5 - u^3) \, du = \frac{1}{6}u^6 - \frac{1}{4}u^4 + C$$

$$= \frac{1}{6}\sec^6 x - \frac{1}{4}\sec^4 x + C$$

You try!

$\int \tan^4(2x)dx$

1) split up the tangent into 2 tangent squares

$$= \int \tan^2(2x)\tan^2(2x)\,dx$$

2) convert one of the $\tan^2(2x)$ to secant squared

$$= \int (\sec^2(2x) - 1)\cdot \tan^2(2x)\,dx$$

3) distribute everything

$$\int \left[\sec^2(2x)\tan^2(2x) - \tan^2(2x)\right] dx$$

4) for the $\tan^2(2x)$ that is left over, convert it to a secant squared

$$\int \left[\sec^2(2x)\tan^2(2x) - (\sec^2(2x) - 1)\right] dx$$

5) integrate!

$$\int \sec^2(2x)\tan^2(2x)\,dx - \int \left[\sec^2(2x) - 1\right] dx$$

$u = \tan(2x)$

$du = 2\sec^2(2x)\,dx$

$$\int \tfrac{1}{2}u^2\,du - \int \left[\sec^2(2x) - 1\right] dx$$

$$\tfrac{1}{6}u^3 - \int \left[\sec^2(2x) - 1\right] dx$$

$$= \tfrac{1}{6}\tan^3(2x) - \tfrac{1}{2}\tan(2x) + x + C$$

Let's sum it up!

- odd powers of sine and/or cosine - pull out one function for your du and convert the rest to the same function, then let u = that function
- even powers of sine and/or cosine - use reduction formula(s) to reduce to an odd power - distribute everything and continue the process with any remaining even powers until everything is either to the first power (integrate) or an odd power (use odd power method above)
- for secants and tangents, if the power of secant is even, pull a squared out for du, convert everything else to tangents, and u-sub
- if your integral is just secant to an odd power, use funky integration by parts

What have we learned?

- Can I evaluate integrals of a variety of functions involving powers of sine, cosine, secant and tangent?