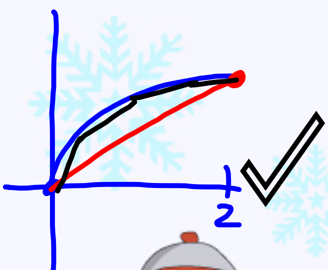
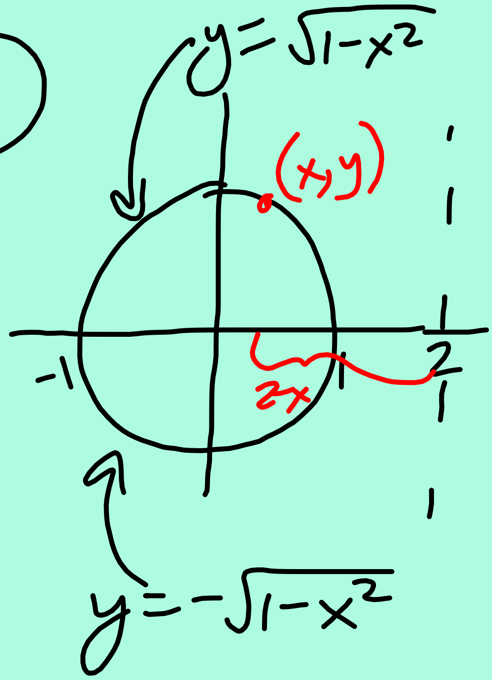
Let's warm up with a warmup!!

Approximate the length of the arc formed by the curve y = $\sqrt{x}$ from x = 0 to x = 2 using any method. (Calculators permitted)



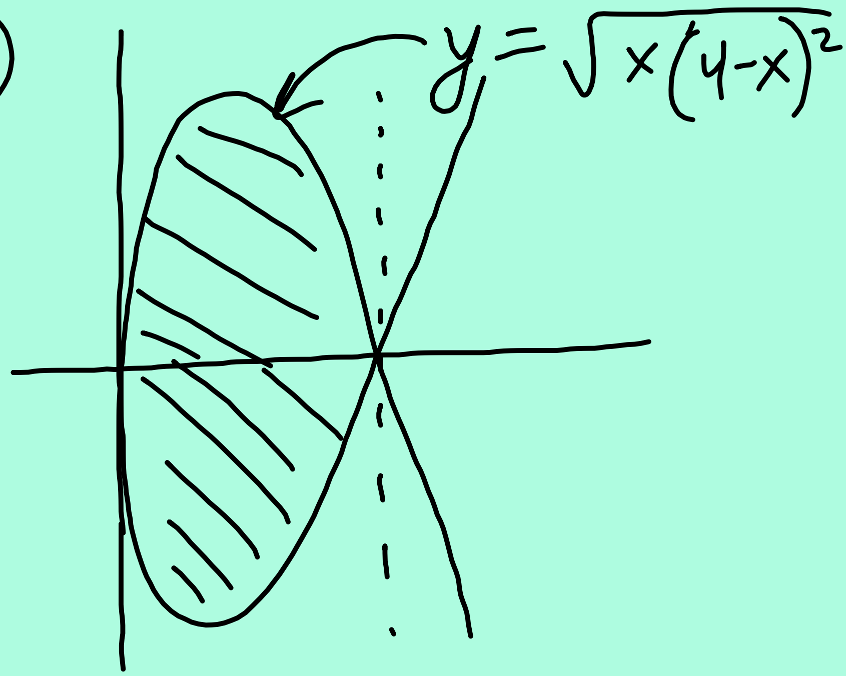
The actual arc length is about 2.562. How close did you get?

(43)



$$V = 2 \cdot 2\pi \int_{-1}^{1} (2-x)\sqrt{1-x^2}\, dx$$

57



a) $V=\pi\int_0^4 x(4-x)^2\,dx$

b) $V=2\cdot 2\pi\int_0^4 x\sqrt{x(4-x)^2}\,dx$

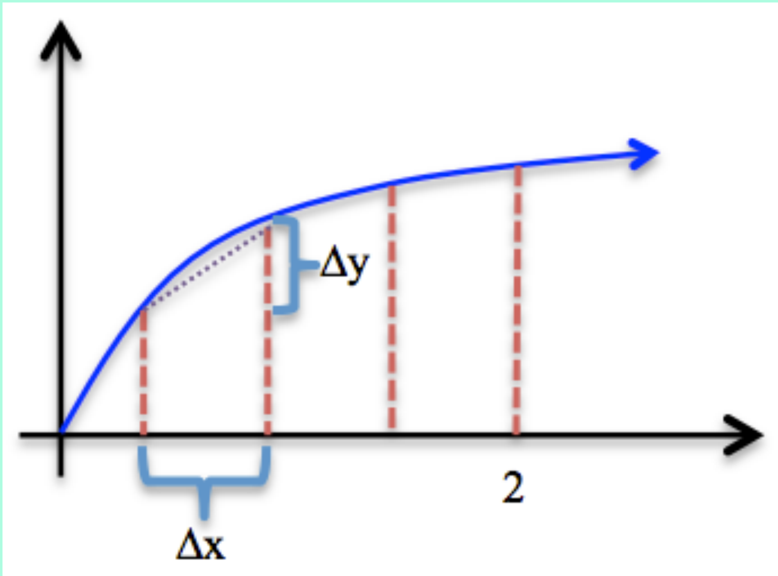
c) $V=2\cdot 2\pi\int_0^4 (4-x)\sqrt{x(4-x)^2}\,dx$

7.4 Arc Length and Surface Area!!

Essential Learning Target:

- Uses definite integrals to calculate the length of a planar curve defined by a function.

Can you derive the formula for arc length?



- Approximate length of single arc: $\sqrt{(\Delta x)^2 + (\Delta y)^2}$
- Simplify to: $\Delta x\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$
- The limit as each of these intervals becomes super small would give us: $dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$
- So to get the entire arc, take the sum of each length to get...

ARC LENGTH:

$$S = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

ex) Find the length of $f(x) = \sqrt{x}$ from x = 0 to x = 2.

✓ $$S = \int_0^2 \sqrt{1 + \left(\frac{1}{2}x^{-\frac{1}{2}}\right)^2}\,dx \approx 2.562$$

You try! (no calculators) :)

1) Find the length of $y = x^3/6 + 1/(2x)$ on $[^1/_2, 2]$

$y' = \frac{1}{2}x^2 - \frac{1}{2x^2} = \frac{x^4-1}{2x^2}$

✓ $S = 33/16$

$AL = \int_{\frac{1}{2}}^{2} \sqrt{1 + \frac{(x^4-1)^2}{4x^4}}\, dx$

$= \int_{\frac{1}{2}}^{2} \sqrt{\frac{4x^4 + x^8 - 2x^4 + 1}{4x^4}}\, dx = \int_{\frac{1}{2}}^{2} \sqrt{\frac{x^8 + 2x^4 + 1}{4x^4}}\, dx$

$= \int_{\frac{1}{2}}^{2} \sqrt{\frac{(x^4+1)^2}{4x^4}}\, dx = \int_{\frac{1}{2}}^{2} \frac{x^4+1}{2x^2}\, dx$

$= \int_{\frac{1}{2}}^{2} \frac{1}{2}x^2 + \frac{1}{2}x^{-2}\, dx$

$= \frac{1}{6}x^3 - \frac{1}{2x}\Big|_{\frac{1}{2}}^{2} = \frac{4}{3} - \frac{1}{4} - \left(\frac{1}{48} - 1\right)$ ☺

$= \frac{64}{48} - \frac{12}{48} - \frac{1}{48} + \frac{48}{48} = \frac{99}{48} = \frac{33}{16}$

2) Find the length of $(y - 1)^3 = x^2$ on $[0, 8]$

(Hint: you will need to solve this one for x in terms of y and integrate with respect to y) (Hint #2: don't forget to find those new y-limits of integration!) :)

$x = (y-1)^{\frac{3}{2}}$ $[1, 5]$

$x' = \frac{3}{2}(y-1)^{\frac{1}{2}}$

$\int_1^5 \sqrt{1 + \left[\frac{3}{2}(y-1)^{\frac{1}{2}}\right]^2}\, dy = \int_1^5 \sqrt{1 + \frac{9}{4}(y-1)}\, dy$

$= \int_1^5 \sqrt{1 + \frac{9}{4}y - \frac{9}{4}}\, dy = \int_1^5 \sqrt{\frac{9}{4}y - \frac{5}{4}}\, dy$

$= \frac{4}{9} \cdot \frac{2}{3}\left(\frac{9}{4}y - \frac{5}{4}\right)^{\frac{3}{2}}\Big|_1^5$

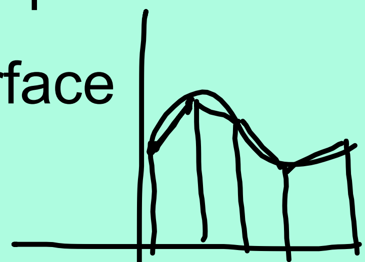
$= \frac{8}{27}(10)^{\frac{3}{2}} - \frac{8}{27}(1)$

✓ $S = (1/27)(40^{3/2} - 4^{3/2})$

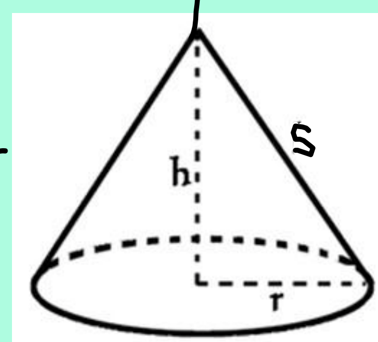
Deriving formulas is so much fun, let's derive another one!

Let's start with a blast from the past.

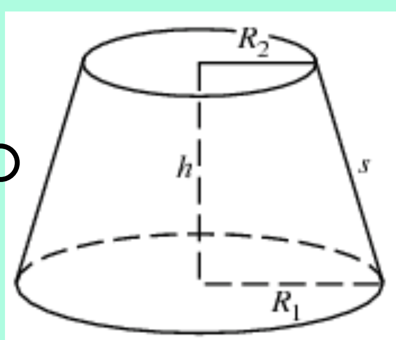
What is the formula for the surface area of a cone?



- $SA = \pi r^2 + \pi rs$



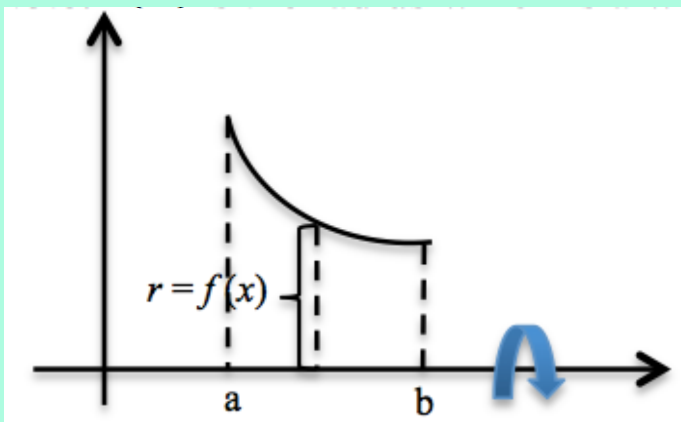
- Dropping the base portion, we get the lateral surface area = πrs

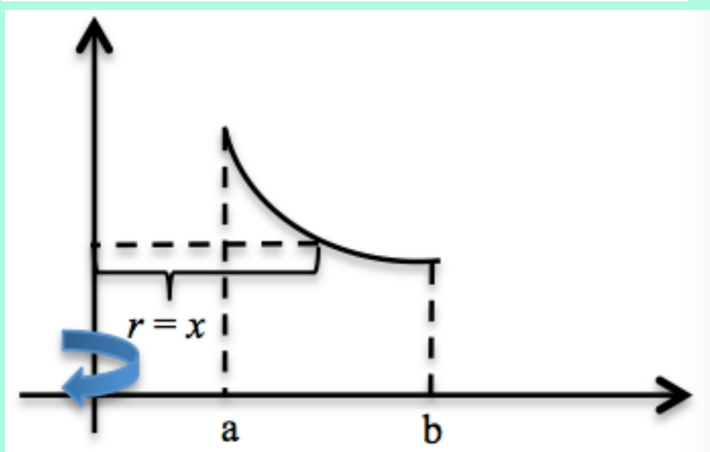


- Based on this, the surface area of the frustrum of a cone is $2\pi Rs$, where R is the average of the two radii on each end of the frustrum

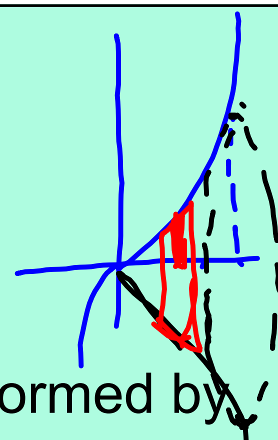
Surface area: $$SA = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\,dx$$

Note: r(x) is always in terms of x, regardless of the orientation of the axis of revolution





$$SA = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\,dx$$

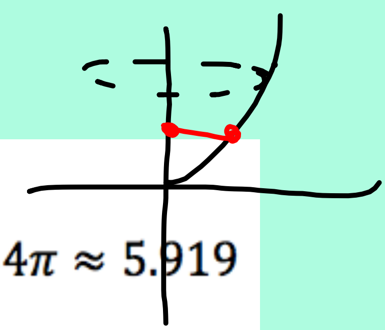


ex) Find the area of the surface formed by revolving the graph of $f(x) = x^3$ on the interval [0, 1]

a) about the x-axis

$$SA = 2\pi \int_0^1 x^3\sqrt{1 + (3x^2)^2}\,dx \approx 1.134\pi \approx 3.563$$

b) about the y-axis

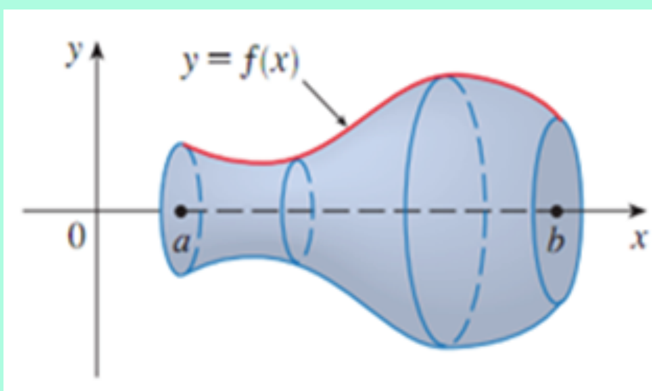


$$SA = 2\pi \int_0^1 x\sqrt{1 + (3x^2)^2}\,dx \approx 1.884\pi \approx 5.919$$

Fun with definitions!!

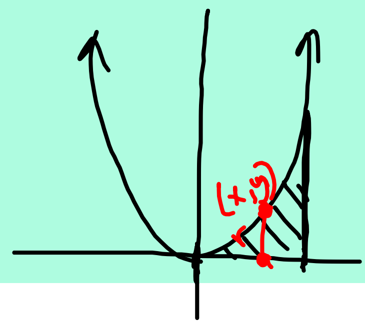
A rectifiable curve is one that has a finite arc length.

If a function is revolved about a line, the resulting surface is called the surface of revolution. Note that this does NOT include the surfaces as though the object formed was a solid. So for the figure below, the circular 'bases' would not be included, they would be treated as empty space.



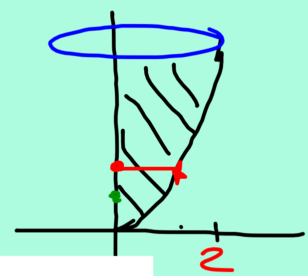
You try! Find the area of the surface formed by revolving the graph of $f(x) = x^2$ on the interval [0, 2]

a) about the x-axis



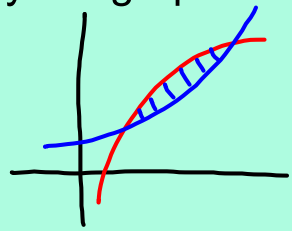
$$SA = 2\pi \int_0^2 x^2\sqrt{1 + (2x)^2}\,dx \approx 16.942\pi \approx 53.225 \text{ or } 53.226$$

b) about the y-axis



$$SA = 2\pi \int_0^2 x\sqrt{1 + (2x)^2}\,dx \approx 11.515\pi \approx 36.176 \text{ or } 36.177$$

You try again! A region is bounded by the graphs of $f(x) = 6x - x^2$ and $g(x) = x^2 - 4x + 8$.
Find the following for this region:



a) the area of the region
b) the volume of the solid formed by revolving the region about the line $y = -2$
c) the volume of the solid formed by revolving the region about the line $x = 6$
d) the length of the top 'arc' of the region
e) the outer surface area of the solid formed by revolving the region about the x-axis

a) ✓ $A = \int_1^4 [(6x - x^2) - (x^2 - 4x + 8)]dx = 9$

b) ✓ $V = \pi \int_1^4 [(6x - x^2 + 2)^2 - (x^2 - 4x + 8 + 2)^2]\, dx = 153\pi$
≈ 480.663 or 480.664

c) ✓ $V = 2\pi \int_1^4 (6 - x)[(6x - x^2) - (x^2 - 4x + 8)]\, dx = 63\pi \approx 197.920$

d) ✓ $S = \int_1^4 \sqrt{1 + (6 - 2x)^2}dx \approx 6.125$ or 6.126

e) ✓ $SA = 2\pi \int_1^4 (6x - x^2)\sqrt{1 + (6 - 2x)^2}dx \approx 92.108\pi$
≈ 289.365 or 289.366

What have we learned?

- Can I calculate the length of an arc?
- Can I calculate the surface area of a solid?

January 10, 2017