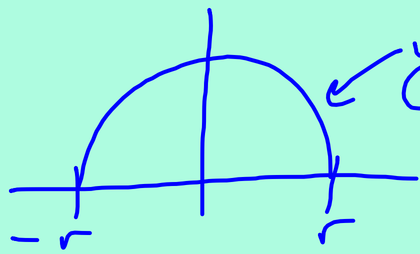
Let's warm up with a warmup!!

Let's try 2014 #2

(49) $V_s = \frac{4}{3}\pi r^3$

$y = \sqrt{r^2 - x^2}$



$$V = \pi \int_{-r}^{r} \left(\sqrt{r^2 - x^2}\right)^2 dx$$

$$= \pi \int_{-r}^{r} (r^2 - x^2)\, dx$$

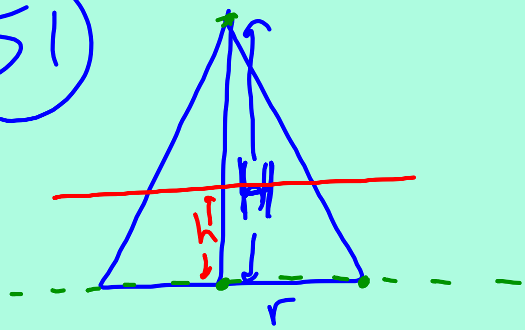
$$= \pi \left[r^2 x - \frac{1}{3}x^3 \Big|_{-r}^{r} \right]$$

$$= \pi \left[r^3 - \frac{1}{3}r^3 - \left(-r^3 + \frac{1}{3}r^3\right) \right]$$

$$= \pi \left[2r^3 - \frac{2}{3}r^3 \right]$$

$$= \frac{4}{3}\pi r^3$$

51



$$y = -\frac{H}{r}x + H$$

$$y - H = -\frac{H}{r}x$$

$$x = -\frac{r}{H}(y - H)$$

$$x = -\frac{ry}{H} + r$$

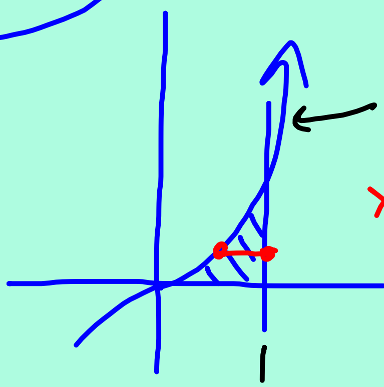
$$V = \pi \int_0^h \left(-\frac{ry}{H} + r\right)^2 dy$$

$$= \pi \int_0^h \left(\frac{r^2y^2}{H^2} - 2\frac{r^2y}{H} + r^2\right) dy$$

$$= \pi \left[\frac{1}{3}\frac{r^2}{H^2}y^3 - \frac{r^2}{H}y^2 + r^2 y \right|_0^h$$

$$= \pi \left[\frac{1}{3}\frac{r^2h^3}{H^2} - \frac{r^2h^2}{H} + r^2 h\right]$$

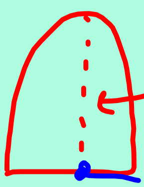
63d



$\leftarrow y = x^3$

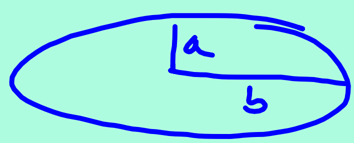
$x = \sqrt[3]{y}$

area of an ellipse



$\leftarrow 2(1-\sqrt[3]{y})$

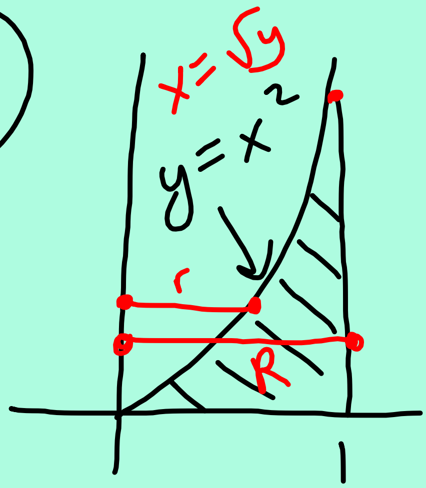
$1-\sqrt[3]{y}$



$= \pi ab$

$$V = \int_0^1 \frac{1}{2}\pi\left(\frac{1}{2}\left(1-\sqrt[3]{y}\right)\right)\left(2\left(1-\sqrt[3]{y}\right)\right)dy$$

71

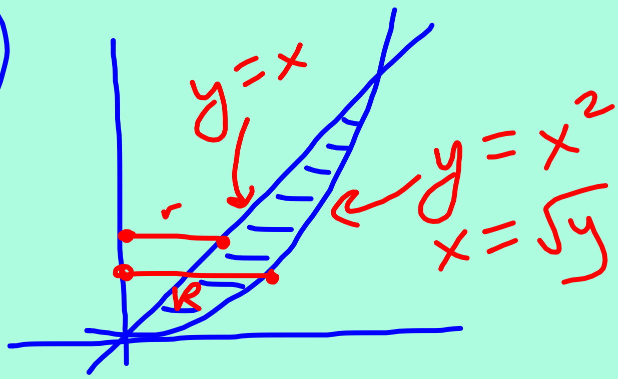


$$R = 1 - 0 = 1$$

$$r = \sqrt{y} - 0 = \sqrt{y}$$

$$V = \pi \int_0^1 \left[1^2 - (\sqrt{y})^2\right] dy$$

73



$R = \sqrt{y} - 0$

$r = y - 0$

$$V = \pi \int_0^1 \left[(\sqrt{y})^2 - y^2 \right] dy$$

$$= \pi \int_0^1 (y - y^2)\, dy$$

$$= \pi \left[\frac{1}{2} y^2 - \frac{1}{3} y^3 \right] \Big|_0^1$$

$$= \pi \left[\frac{1}{2} - \frac{1}{3} \right] = \frac{1}{6} \pi$$

7.3 Volumes by Shells!!

At the end of this lesson you will be able to:

- Calculate the volume of a solid of revolution using the shell method

Why use shells? Sometimes the volume of a solid of revolution just can't be easily found using disks or washers. For example: suppose we wanted the volume of the solid formed by revolving $y = (x - 1)(x - 3)^2$ about the y-axis. Because our axis is vertical, this would require us to solve for x in terms of y. Good luck with this! Good news, there is an alternative and it's not really any more difficult than disks or washers.

What on earth do these shapes look like?

Let's see!

Volume by shells: the surface area of each shell would be $2\pi rh$. If we multiply this by a 'thickness' (and take the limit as these 'thicknesses' approach 0 (dx or dy)) and then we take the infinite sum of these 'surface areas' we get the volume of the figure.

r = radius from the axis of rotation to x (or y)

h = the height of each shell

vertical axis => dx

horizontal axis => dy

$$V = 2\pi \int_a^b rh\,(dx \text{ or } dy)$$

What are the radius and height of each shell?

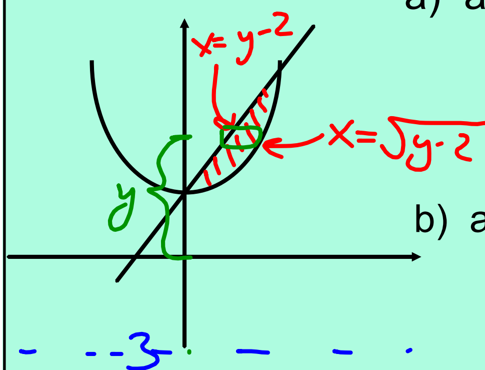
The radius is the distance from the axis to a point (x, y) on the region

axis and x $\Rightarrow dx$
axis and y $\Rightarrow dy$

The height is the space between the boundaries (top - bottom) or (right - left)

ex) Let's revolve a region bounded by $y = x^2 + 2$ and $y = x + 2$ $\rightarrow x = y - 2$

$x = \pm\sqrt{y-2}$



a) about x-axis

radius = y

height = $\sqrt{y - 2} - (y - 2)$

b) about y = -3

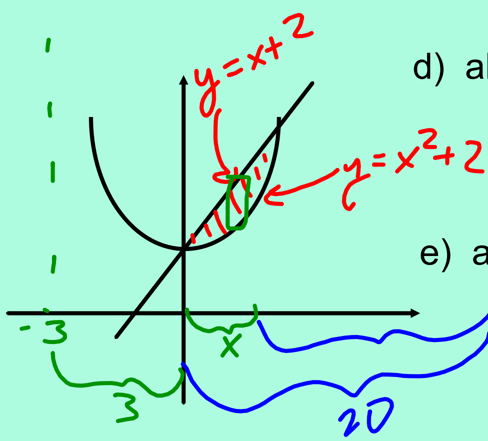
radius = y - -3 = y + 3

height = $\sqrt{y - 2} - (y - 2)$

c) about y = 20

radius = 20 - y

height = $\sqrt{y - 2} - (y - 2)$



d) about y-axis

radius = x

height = $x + 2 - (x^2 + 2)$

e) about x = -3

radius = x - -3 = x + 3

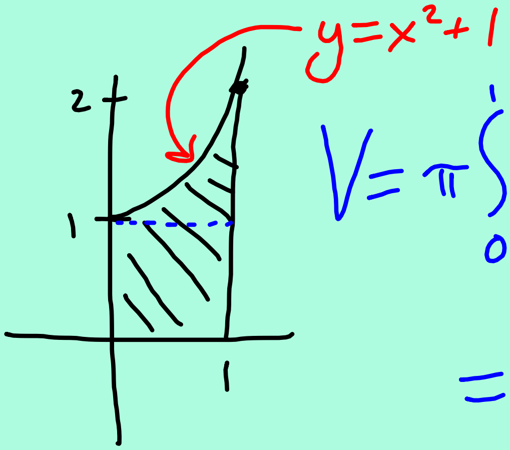
height = $x + 2 - (x^2 + 2)$

f) about x = 20

radius = 20 - x

height = $x + 2 - (x^2 + 2)$

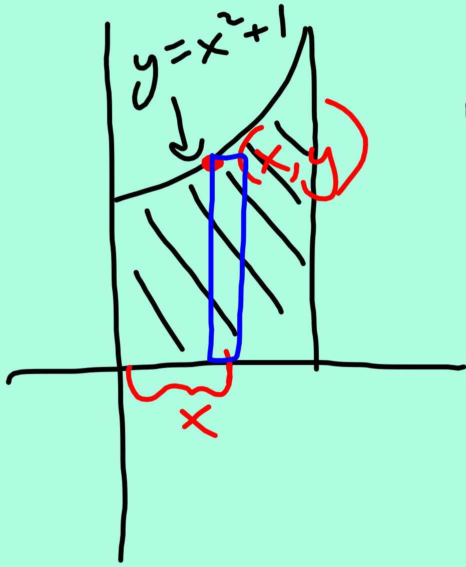
2nd warmup) Use disks/washers to find the volume of the solid formed by revolving the region bounded by $y = x^2 + 1$, $y = 0$, $x = 0$ and $x = 1$ about the y-axis.



$$V = \pi\int_0^1 1^2\,dy + \pi\int_1^2 \left(1^2 - \left(\sqrt{y-1}\right)^2\right) dy$$

$$= \frac{3}{2}\pi$$

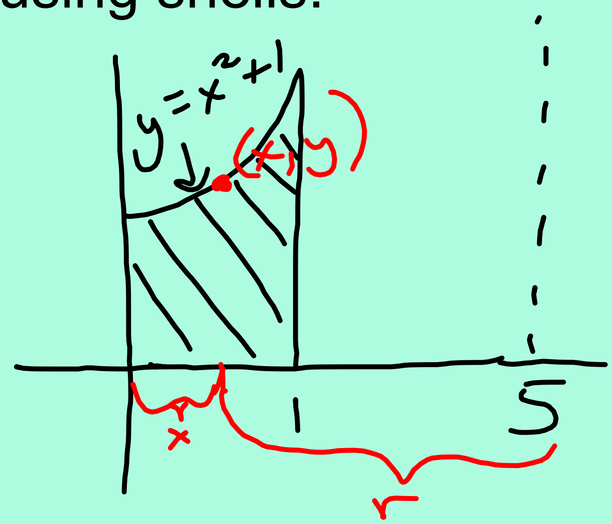
ex) Let's do the same thing using shells.



$$V = 2\pi\int_0^1 x\left(x^2+1\right) dx$$

$$= \frac{3}{2}\pi$$

What about a different axis? Suppose we revolve the region bounded by $y = x^2 + 1$, $x = 0$, $x = 1$ and $y = 0$ about $x = 5$. Find the volume using shells.



$$V = 2\pi \int_0^1 (5-x)(x^2+1)\,dx$$

Which to pick?

Here's the breakdown (just my preference):

Disk method: use when the region borders the axis of rotation (no space in between) and the function is defined in terms of the variable you want (function is in terms of x with a horizontal axis or function is in terms of y with a vertical axis)

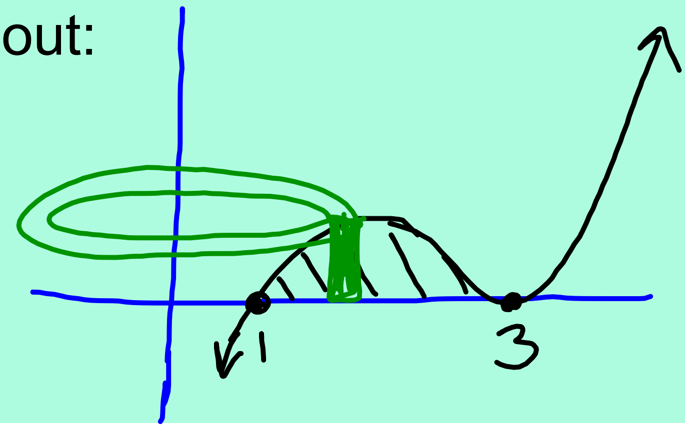
Washer method: use same as disk except when there is space between the region and the axis of rotation

Shell method: use whenever the variable is 'opposite' the axis (input is x and axis is vertical or input is y and axis is horizontal)

You try! Find the volume of the solid formed by revolving the region bounded by $y = (x - 1)(x - 3)^2$ and the x-axis about:

a) the y-axis

b) x = -4



$$V = 2\pi \int_1^3 x(x-1)(x-3)^2 dx = \frac{24}{5}\pi = 4.8\pi \approx 15.079 \text{ or } 15.080$$

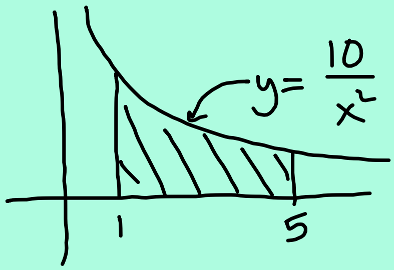
$$V = 2\pi \int_1^3 (x+4)(x-1)(x-3)^2 dx = \frac{232}{15}\pi$$

$$\approx 15.466\pi \approx 15.467\pi \approx 48.589 \text{ or } 48.590$$

Let's mix it up! Solve the following using any method(s).

Find the volume of the solid formed by revolving the region bounded by $y = 10/x^2$, $y = 0$, $x = 1$ and $x = 5$ about:

a) the x-axis
b) the y-axis
c) $y = 10$
b) $x = -3$



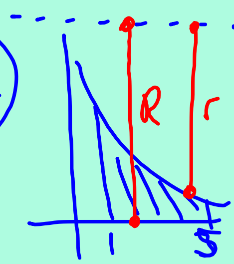
a) Disk

$$V = \pi \int_1^5 \left(\frac{10}{x^2}\right)^2 dx$$

b) Shell

$$V = 2\pi \int_1^5 x\left(\frac{10}{x^2}\right) dx$$

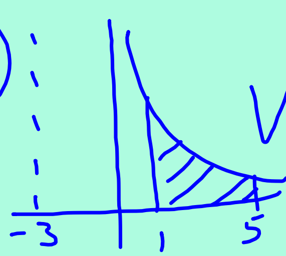
c)



$R = 10 - 0$
$r = 10 - \frac{10}{x^2}$

$$V = \pi \int_1^5 \left[10^2 - \left(10 - \frac{10}{x^2}\right)^2\right] dx$$

b2)



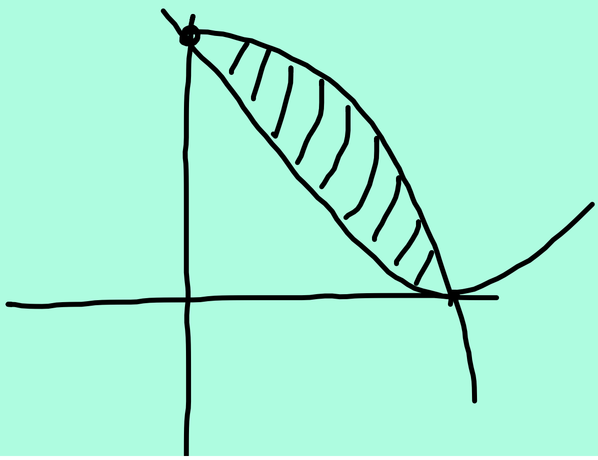
$$V = 2\pi \int_1^5 (x+3)\left(\frac{10}{x^2}\right) dx$$

One more! Solve the following using any method(s).

Find the volume of the solid formed by revolving the region bounded by y = 4cosx and y = $(x - 2)^2$ about:

a) y = -3

b) x = 5



$$V = \pi \int_{0}^{1.5109741} [(4\cos x - -3)^2 - ((x-2)^2 - -3)^2]dx$$

$$\approx 14.361\pi \; or \; 14.362\pi \approx 45.118 \; or \; 45.119$$

$$V = 2\pi \int_{0}^{1.5109741} [(5-x)(4\cos x - (x-2)^2)]\,dx \approx 11.646\pi \approx 36.588$$

What have we learned?

- Can I find the volume of a solid using the shell method?
- Do I know which method is best to use for each type of problem?

January 09, 2017