WARMUP!!

Find the inverse, $f^{-1}(x)$, for each of the following:

1) $f(x) = 6x + 2$

2) $f(x) = \sqrt{x - 5}$

3) $f(x) = (x - 2)^3 - 3$

① $x = 6y + 2$

$y = \frac{x-2}{6}$

② $x = \sqrt{y-5}$

$y = x^2 + 5$

③ $x = (y-2)^3 - 3$

$y = \sqrt[3]{x+3} + 2$

5.3 Inverse Functions!

At the end of this lesson you will be able to:

> Determine if 2 functions are inverses
> Find the inverse of a function
> State the domain/range of inverse functions
> Determine if a function will have an inverse that is a function
> Determine if a function is one-to-one
> Sketch graphs of inverse functions
> Define 'monotonocity'

Warmup #2: In your groups, make a list of everything you can think of related to inverse functions.

- x's and y's are swapped
- domains and ranges are swapped
- graphs are symmetric over y = x
- f(x) must be one-to-one for the inverse to be a function passes the horizontal line test
- if f(x) and g(x) are inverses, then f(g(x)) = x and g(f(x)) = x

A few helpful definitions:

A function is **one-to-one** if there is exactly one x-value for each y-value (i.e. if the function passes the horizontal line test).

A function is **monotonic** if it is one-directional, meaning always increasing or always decreasing. A monotonic function will always be one-to-one and therefore will always have an inverse function.

(What is an easy way to determine if a function is monotonic?) f' sign line

Note, the domain of an inverse must be restricted to equal the range of the original function

ex) If $f(x) = \sqrt{x + 3} - 4$, find $f^{-1}(x)$ and state its domain

R: $[-4, \infty)$

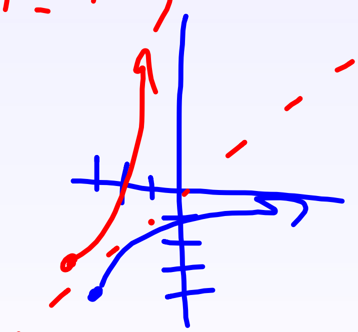
$x = \sqrt{y+3} - 4$

$x + 4 = \sqrt{y+3}$

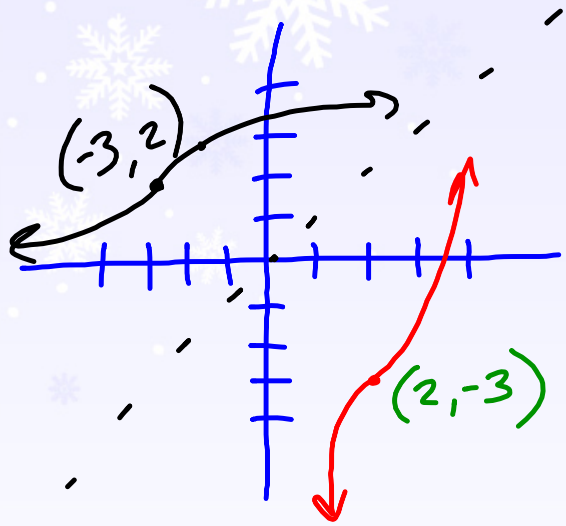
$y + 3 = (x+4)^2$

$y = (x+4)^2 - 3$

D: $[-4, \infty)$



Graph $y = (x - 2)^3 - 3$ and its inverse



DERIVATIVE of an Inverse!!

Let's derive the formula!

Suppose f(x) and g(x) are inverse functions.

Then $f(g(x)) = x$.

Differentiate both sides: $f'(g(x)) \cdot g'(x) = 1$

Solve for g'(x): $g'(x) = \dfrac{1}{f'(g(x))}$

So $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$

There are 2 methods for finding the derivative of an inverse at x = a.

ex) $f(x) = x^3 + 4x - 2$, find $(f^{-1})'(3)$

Use the formula

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

1) find $f^{-1}(a)$

(set f(x)=a, solve for x)

if 3 is an x-value on f^{-1}, then 3 is a y-value on f

$x^3+4x-2=3$ → use guess + check

$x=1$

2) find $f'(x)$

$f'(x)=3x^2+4$

3) find $f'(f^{-1}(a))$

$f'(1)=7$

4) flip it $\frac{1}{7}$

Swap x and y

Differentiate implicitly

1) find $f^{-1}(a)$

(set f(x)=a, solve for x)

$3 = x^3+4x-2$

$x=1 \quad f(1)=3$

so $f^{-1}(3)=1$

2) Swap x and y

$x = y^3+4y-2$

3) Differentiate both sides

$1 = 3y^2\frac{dy}{dx}+4\frac{dy}{dx}$

4) Plug in the x and y values

(be careful!)

$\frac{dy}{dx} = \frac{1}{3y^2+4}$

$\left.\frac{dy}{dx}\right|_{y=1} = \frac{1}{7}$

You try! Suppose $f(x) = \sqrt{x^3 - 7}$, find $(f^{-1})'(1)$ (without finding the inverse function)

$1 = \sqrt{x^3-7}$ so $x=2$

$$f'(x) = \frac{1}{2}(x^3-7)^{-\frac{1}{2}}(3x^2)$$

$$= \frac{3x^2}{2\sqrt{x^3-7}}$$

$$f'(2) = \frac{12}{2\sqrt{1}} = 6$$

$$\frac{1}{6}$$

$$x = \sqrt{y^3-7}$$

$$1 = \frac{1}{2}(y^3-7)^{-\frac{1}{2}}(3y^2)\frac{dy}{dx}$$

$$1 = \frac{1}{2}(1)^{-\frac{1}{2}}(12)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{6}$$

Let's try an AP example!

2007 #3 (should look familiar)

Review!

Suppose $f(x) = \dfrac{x}{x^2+4}$. Approximate the area between the curve and the x-axis from 0 to 4 using each of the following (make a table first). Then find the exact area. (Calculators permitted)

a) 4 left-handed rectangles

b) 4 right-handed rectangles

c) 2 midpoint rectangles

d) 4 trapezoids

x	0	1	2	3	4
f(x)					

a) 1[0 + 1/5 + 1/4 + 3/13] = 177/260 ≈ 0.681

b) 1[1/5 + 1/4 + 3/13 + 1/5] = 229/260 ≈ 0.881

c) 2[1/5 + 3/13] = 56/65 ≈ 0.862

d) (1/2)(1)[(0 + 1/5) + (1/5 + 1/4) + (1/4 + 3/13) + (3/13 + 1/5)] = 203/260 ≈ 0.781

e) (4/12)[0 + 4(1/5) + 2(1/4) + 4(3/13) + 1/5] = 21/26 ≈ 0.808

exact area =

$$\int_0^4 \frac{x}{x^2+4}\,dx = \frac{1}{2}\int_4^{20} \frac{1}{u}\,du = \frac{1}{2}\ln|u|\Big|_4^{20} = \frac{1}{2}(\ln 20 - \ln 4) = \frac{1}{2}\ln 5 \approx 0.805$$

Review!! (NO CALCULATORS!!)

1) Find dy/dx if

a) $y = \ln(7x)$

b) $y = \ln(x^3)$

c) $y = \ln(x - 3)$

d) $y = \ln(2x^3 + x)$

e) $y = \ln(\cos x)$

f) $y = \ln(\sin x)$

g) $y = \ln\sqrt{x^2 + 1}$

h) $y = \ln(1/(2x-1))$

2) Use logarithms to find dy/dx if

a) $y = \sqrt{x^2 + 1}$

b) $y = \dfrac{(x + 1)(x + 2)}{(x + 3)}$

1) a) $1/x$ b) $3/x$ c) $1/(x - 3)$

d) $(6x^2 + 1)/(2x^3 + x)$ e) $-\tan x$

f) $\cot x$ g) $x / (x^2 + 1)$ h) $-2/(2x - 1)$

2a) $(x\sqrt{x^2 + 1}) / (x^2 + 1) = x / \sqrt{x^2 + 1}$

2b) $\left(\dfrac{1}{x + 1} + \dfrac{1}{x + 2} - \dfrac{1}{x + 3}\right)\dfrac{(x + 1)(x + 2)}{x + 3}$

What have we learned?

- Can I find an inverse function?
- Can I use calculus to determine if a function will have an inverse function?
- Can I state the domain of an inverse?
- Can I find the derivative of an inverse at a specific value of x?