WARM UP!!

1. What does $\dfrac{f(x+h)-f(x)}{h}$ represent?
2. If f(x) = x^2 - 2x, what does f '(3) equal?
3. What does $\dfrac{d}{dx}(-3\cos x)$ equal?
4. Define 'derivative'.
5. Evaluate $\lim_{x \to 0^+} \dfrac{1}{x}$
6. What is the product rule for derivatives?
7. What does d/dx (7) equal and why?
8. Give an equation for a function that is not differentiable over its entire domain.
9. What is the relationship between the formula for the position of an object and the formula for the velocity of the object?
10. If f(x) has a maximum value at x = c, what do we know about f '(c)?

Let's check! ✓

1. Slope of the secant line between the points (x, f(x)) and (x + h, f(x + h))

2. f '(3) = 4

3. 3sinx

4. A derivative is a function whose output is the slope of the line tangent to a function at any given value of x

5. ∞

6. $\frac{d}{dx}(f \bullet g) = f'(x)g(x) + f(x)g'(x)$

7. The derivative of 7 is zero because the graph of the constant function, y = 7, is a horizontal line so it's slope would always be zero.

8. Any function that contains a sharp turn or a vertical tangent would not be differentiable on it's domain. (Easiest example that is not piecewise is $y = x^{2/3}$.)

9. The velocity function would be the derivative of the position function.

10. f '(c) must either be 0 or undefined.

4.2b Exact Area with Limits!

Essential Learning Target:

Translate the information in a definite integral into the limit of a related Riemann sum, and the limit of a Riemann sum can be written as a definite integral

Bernhard Riemann

1826 - 1866

Bernhard Riemann was a German mathematician (mentored by Gauss). His father was a Lutheran minister and he went to school to study theology, but Gauss convinced him to change his major to mathematics. He was most famous for describing the world in dimensions higher than 3, and is the reason I had to struggle through nth dimensional differential geometry when I was 18 (ughh!). For us, he was able to formalize the use of sums of areas of rectangles for area approximation and then extend this using limits to eventually lead to the creation of the definite integral, which is the main link between geometry and calculus.

BTW, when he died of tuberculosis, his housekeeper tossed the majority of his papers which included stacks of unpublished work.

Let's see what's really going on! Can we derive the formula for exact area?

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{b-a}{n}\right) \cdot f\left(a + \frac{b-a}{n} \cdot i\right)$$

$$\text{width} \cdot f\left(\text{left endpoint} + \text{width} \cdot i\right)$$

ex) Find the exact area between $y = 4 - x^2$ and the x-axis from [-2, 2].

$$\text{width} = \frac{b-a}{n} = \frac{2 - -2}{n} = \frac{4}{n}$$

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4}{n}\right)\left[4 - \left(-2 + \frac{4i}{n}\right)^2\right]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4}{n}\right)\left[4 - \left(4 - \frac{16i}{n} + \frac{16i^2}{n^2}\right)\right]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4}{n}\right)\left[\frac{16i}{n} - \frac{16i^2}{n^2}\right]$$

$$= \lim_{n\to\infty} \left(\frac{4}{n}\right) \sum_{i=1}^{n} \left[\frac{16i}{n} - \frac{16i^2}{n^2}\right]$$

$$= \lim_{n\to\infty} \left(\frac{4}{n}\right)\left[\frac{16}{n}\sum_{i=1}^{n} i - \frac{16}{n^2}\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty} \left(\frac{4}{n}\right)\left[\frac{16}{n}\cdot\frac{n(n+1)}{2} - \frac{16}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{\overset{2}{\cancel{4}}\cdot 16}{\cancel{2}} - \frac{4\cdot 16\cdot \cancel{2}}{\underset{3}{\cancel{6}}} = 32 - \frac{64}{3} = \boxed{\frac{32}{3}}$$

You try! Find the exact area of $f(y) = y^3 + 1$ on $[1, 2]$.

$a = 1$, $b = 2$

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{n}\right)\left[\left(1 + \frac{1}{n}\cdot i\right)^3 + 1\right]$$

$$= \lim_{n\to\infty} \left(\frac{1}{n}\right) \sum_{i=1}^{n} \left[1 + \frac{3i}{n} + \frac{3i^2}{n^2} + \frac{i^3}{n^3} + 1\right]$$

$$= \lim_{n\to\infty} \left(\frac{1}{n}\right) \sum_{i=1}^{n} \left[2 + \frac{3i}{n} + \frac{3i^2}{n^2} + \frac{i^3}{n^3}\right]$$

$$= \lim_{n\to\infty} \left(\frac{1}{n}\right) \left[\sum_{i=1}^{n} 2 + \frac{3}{n}\sum_{i=1}^{n} i + \frac{3}{n^2}\sum_{i=1}^{n} i^2 + \frac{1}{n^3}\sum_{i=1}^{n} i^3\right]$$

$$= \lim_{n\to\infty} \left(\frac{1}{n}\right) \left[2n + \frac{3}{n}\cdot\frac{n(n+1)}{2} + \frac{3}{n^2}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{1}{n^3}\cdot\frac{n^2(n+1)^2}{4}\right]$$

$$= 2 + \frac{3}{2} + \frac{6}{6} + \frac{1}{4} = \boxed{\frac{19}{4}}$$

You try! Find the exact area of $f(y) = y^3 + 1$ on $[1, 2]$.

$a \quad b$

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{n}\right)\left[\left(1 + \frac{1}{n}\cdot i\right)^3 + 1\right]$$

$\frac{2}{1}$

$\frac{ }{4}$

1997 #24

The expression $\frac{1}{50}\left(\sqrt{\frac{1}{50}}+\sqrt{\frac{2}{50}}+\sqrt{\frac{3}{50}}+\ldots+\sqrt{\frac{50}{50}}\right)$ is a Riemann sum approximation for:

a) $\int_0^1 \sqrt{\frac{x}{50}}\,dx$ b) $\int_0^1 \sqrt{x}\,dx$ c) $\frac{1}{50}\int_0^1 \sqrt{\frac{x}{50}}\,dx$ d) $\frac{1}{50}\int_0^1 \sqrt{x}\,dx$ e) $\frac{1}{50}\int_0^{50} \sqrt{x}\,dx$

What have we learned??

- What is the formula for calculating exact area under a curve?