WARM UP!!

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	−4	4	2
4	−1	3	6	7

2007 #3) The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by h(x) = f(g(x)) - 6.

a) Explain why there must be a value r for 1 < r < 3 such that h(r) = -5.

$h(1) = 3$, $h(3) = -7$

$h(3) \le -5 \le h(1)$ ∴ by the IVT, there must be some r in (1,3) such that $h(r) = -5$

b) Explain why there must be a value c for 1 < c < 3 such that h'(c) = -5.

h(x) is diff and therefore cont. on (1,3)

$h(1) = f(g(1)) - 6 = 9 - 6 = 3$

$h(3) = f(g(3)) - 6 = -1 - 6 = -7$

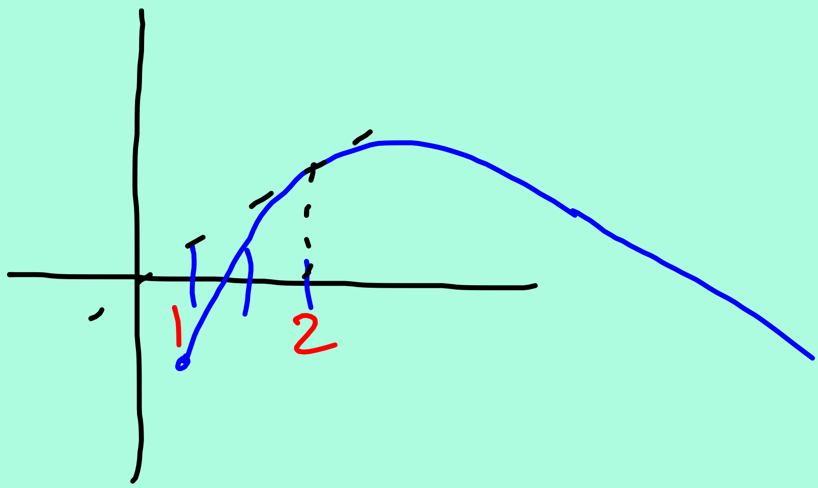
$SOS = \frac{-7-3}{3-1} = -5$

∴ by the MVT there must be some c in (1,3) such that $h'(c) = -5$

7 $f(x) = 3\sqrt{x-1} - x \qquad x \geq 1$

$(1, -1)$

$(2, 1)$



$$f'(x) = 3\left(\tfrac{1}{2}\right)(x-1)^{-\frac{1}{2}} - 1$$

$$= \frac{3}{2\sqrt{x-1}} - 1 \qquad f'(2) = \frac{1}{2}$$

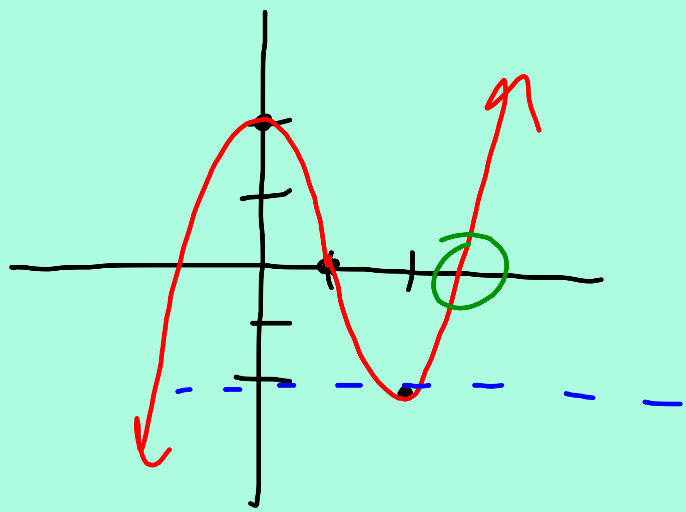
$$y - 1 = \tfrac{1}{2}(x-2)$$

$$y = \tfrac{1}{2}x$$

$$0 = \tfrac{1}{2}x$$

$$x = 0$$

ex) $y = x^3 - 3x^2 + 2$



$(2,-2)$

pt $(3,2)$

$y' = 3x^2 - 6x$ let $x_0 = 3$

$y'(3) = 27 - 18 = 9$

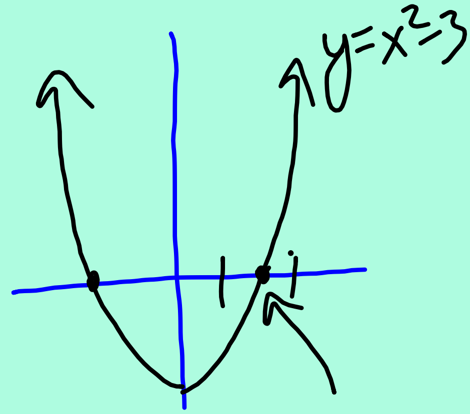
$y - 2 = 9(x - 3)$

$-2 = 9x - 27$

$9x = 25$

$x_1 = \frac{25}{9} \approx 2.78$ actual zero ≈ 2.73

$\sqrt{3} \approx ?$



$y = x^2 - 3$

$(1, -2)$

pt $(2, 1)$

$y' = 2x$

$y'(2) = 4$

$x_0 = 2$

$y - 1 = 4(x - 2)$

$-1 = 4x - 8$

$4x = 7$

$x_1 = \frac{7}{4}$

pt $\left(\frac{7}{4}, \frac{49}{16} - \frac{48}{16}\right)$

$= \left(\frac{7}{4}, \frac{1}{16}\right)$

$m = y'\left(\frac{7}{4}\right) = \frac{7}{2}$

$\cancel{y} - \frac{1}{16} = \frac{7}{2}\left(x - \frac{7}{4}\right)$

$-\frac{1}{56} = x - \frac{7}{4}$

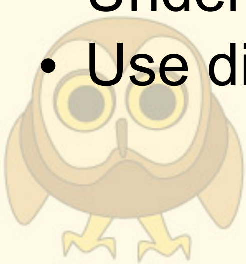
$x = \frac{7}{4} - \frac{1}{56} = \frac{98 - 1}{56}$

$= \frac{97}{56}$

3.9 Differentials!

At the end of this lesson you will be able to:

- Understand what a tangent line approximation is and how it relates to a function value
- Understand what is meant by a 'differential'
- Use differentials to approximate error



Let's see what the 'linearization' of a function really means. Suppose $y = x^2$ and we compare this to the line tangent to y at x = 2.

This is the concept on which tangent line approximation is based.

Tangent line approximation (a.k.a. linear approximation), simply means using y-values on the tangent line equation to approximate y-values on the original function at x-values 'near' the point of tangency.

Write the equation of the line tangent to f(x) at (c, f(c)).

✓ $y - f(c) = f\,'(c)(x - c)$

$x - c$ is denoted as $\triangle x$

and $y - f(c)$ is denoted as $\triangle y$.

So we can rewrite the equation as:

$$\triangle y = f\,'(c)\triangle x$$

When $\triangle x$ is small (approaching zero), we call $\triangle x$, dx, which is called the 'differential' of x.

Similarly, as $\triangle x$ approaches zero, $\triangle y$ becomes known as dy, the differential of y.

So now, $\triangle y = f\,'(c)\triangle x$

becomes $dy = f\,'(x)\,dx$

(think $dy = dy/dx * dx$)

Whoop-de-do! Isn't this just saying the same thing as $dy/dx = f'(x)$, so $dy = f'(x)\,dx$?

All you did was multiply both sides by dx ... big deal!

Yup! But it means so much more!

Differentials open up an entire new world of applications that can be solved using derivatives (and integrals).

Our focus in this section is just an in-depth study of the differentials themselves.

Let's start with a simple new way of looking at derivatives.

ex) Suppose $y = 8x^2 - 7x + 2$, find dy.

$$dy = (16x - 7)\,dx$$

You try! Find dy for the following:

a) $y = 3x\sqrt{x^3 - 4x}$

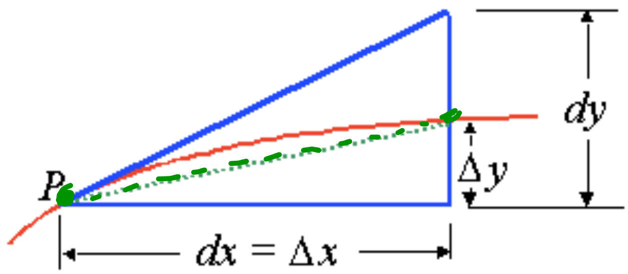
$dy = \left[3\sqrt{x^3-4x} + 3x\left(\frac{1}{2}\right)(x^3-4x)^{-\frac{1}{2}}(3x^2-4)\right]dx$

$= \left[3\sqrt{x^3-4x} + \frac{3x(3x^2-4)}{2\sqrt{x^3-4x}}\right]dx$

b) $y = x\sec^3(5x) = x\left[\sec(5x)\right]^3$

$dy = \left[\sec^3(5x) + x(3)\sec^2(5x)\cdot\sec(5x)\tan(5x)(5)\right]dx$

$= \left[\sec^3(5x) + 15x\sec^3(5x)\tan(5x)\right]dx$



Remember that $\frac{\Delta y}{\Delta x}$ represents <u>slope of a secant line</u> that connects 2 points on a function.

dy/dx represents <u>slope of a line tangent</u> to a function at a specific point.

To compare Δy and dy, think like this:

$$dy = f'(x)\,dx$$

$$\Delta y = f(x + \Delta x) - f(x)$$

ex) Compare Δy and dy for $y = 3x^2 + 2$ at $x = 1$ if $\Delta x = dx = 0.1$.

$$\Delta y = 3(1.1)^2 + 2 - (3(1)^2 + 2)$$
$$= 0.63$$

$$dy = 6x\,dx$$
$$= 6(1)(.1) = .6$$

2nd Warmup: My company produces squares (they're really awesome). The squares are supposed to have a side length of 4", and we promise that the length will never be 'off' by more than 0.01". What is the worst possible scenario for error in the area of our squares?

worst − best

propagated error $= \Delta y = (4.01)^2 - 4^2 \approx .0801$

$A = x^2 \quad dA = 2x\,dx$

estimated propagated error $= dA = 2(4)(.01) = .08$

$$\%\text{ error} = \frac{dA}{A} = \frac{.08}{4^2} \approx .005$$

$= .5\%$

3rd Warmup: We have done so well with our squares, that we've moved on to circles! The radius of our circles is 6" and with an estimated error of no greater than 0.02". What is the worst possible scenario for error in the area of our circles?

propagated error (P.E.) $= \Delta y = \pi(6.02)^2 - \pi(6)^2$

$\approx .755238$

$A = \pi r^2 \quad dA = 2\pi r\, dr$

$EPE = dA = 2\pi(6)(.02) \approx .75398$

$\%\text{error} = \frac{.75398}{36\pi} \approx .0067$

$.667\%$

4th Warmup: Wow, the squares and circles are doing great! So great that now we're making spheres! The spheres have a radius of 5" which will never actually be off by more than .03". What is the worst possible scenario for error in the volume of our spheres?

$V = \frac{4}{3}\pi r^3$ $\quad dV = 4\pi r^2 dr$

$P.E. = \Delta y = \frac{4}{3}\pi(5.03)^3 - \frac{4}{3}\pi(5)^3$

$\approx 9.4814 \text{ in}^3$

$EPE = 4\pi(5)^2(.03) \approx 9.424778$

$\% \text{ error} = \frac{9.424778}{\frac{4}{3}\pi(5)^3} \approx .018$

$= 1.8\%$

Error Propogation!!

Differentials are often used in the approximation of error in measuring devices. In order to do this, let

x = measured value

$x + \Delta x$ = exact value

So Δx = error in measurement

If x is used to compute another value, than the difference between $f(x)$ and $f(x + \Delta x)$ is called the **propagated error**.

Prop. error = $\Delta y = f(x + \Delta x) - f(x)$

Summary of error formulas:

✓**propagated error** = $\Delta y = f(x + \Delta x) - f(x)$

✓**estimated propagated error** = $dy = f'(x)dx$

✓**% error** = $dy / f(x)$

A company produces cylinders whose heights are twice their radii. If the radius of the produced cylinder is 10 inches with an error of 0.25 inches:

a) What is the propagated error of the volume?

b) What is the estimated propagated error of the volume?

c) Using the answer from b), what is the percent error of the volume?

$$V = \pi r^2 h = \pi r^2 (2r) = 2\pi r^3$$

a) $PE = 2\pi(10.25)^3 - 2\pi(10)^3$

≈ 483.11804

b) $dV = 6\pi r^2 dr$

$EPE = 6\pi(10)^2(.25) = 471.238898$

c) $\%\ \text{error} = \dfrac{471.238898}{2\pi(10)^3} = .075$

$= 7.5\%$

The Big Fuzzy Dice Company is producing dice in the shapes of cubes with side lengths of 7". They have estimated an error of around 0.25" in the lengths of the sides of their cubes.

a) Find the propagated error, estimated propagated error, and percent error for the volume of each die

b) Find the propagated error, estimated propagated error, and percent error for the surface area of each die

a) $V = x^3$ $\quad PE = (7.25)^3 - 7^3 \approx 38.078125$

$dV = 3x^2 dx$ $\quad EPE = 3(7)^2(.25) \approx 36.75$

$\%\text{error} = \frac{36.75}{7^3} \approx .107142$

or 10.714%

b) $S = 6x^2$ $\quad PE = 6(7.25)^2 - 6(7)^2 = 21.375$

$dS = 12x\,dx$ $\quad EPE = 12(7)(.25) = 21$

$\%\text{error} = \frac{21}{6(7)^2} \approx .0714286$

or 7.143%

What have we learned??

- Can dy/dx really be separated into dy and dx where each part has its own meaning?
- What is meant by propagated error, estimated propagated error, and percent error? Can I keep all of those terms straight?