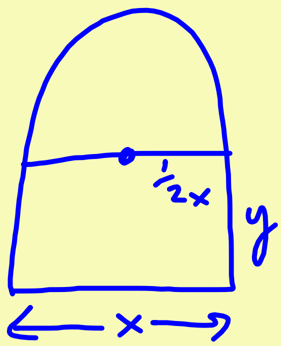
WARMUP!!

Without using a calculator, use any method to approximate the zero(s) of $f(x) = x^5 + x - 1$

✓ A quick check of the derivative will show that the function always has a positive slope and is therefore always increasing, so there can be at most one zero. Simple guess and check shows that $f(0) < 0$ and $f(1) > 0$, so the zero falls in this interval. $f(1/2) < 0$ which narrows it down a bit. The actual zero is around $x = 3/4$ ($x \approx 0.75487767$).



$$A = xy + \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2$$

$$P = x + 2y + \pi\left(\frac{x}{2}\right) = 16$$

$$y = \frac{16 - x - \frac{\pi x}{2}}{2} = \boxed{8 - \frac{1}{2}x - \frac{\pi x}{4}}$$

3.8 Newton's Method!

At the end of this lesson you will be able to:

- Use Newton's Method to approximate zeros of functions

Who is Newton?

Isaac Newton was an English physicist and mathematician in the later half of the 17th century. He is most known for his work with the laws of motion and gravity, as well as the development of calculus. He greatly advanced the study of planetary motion, and was able to prove with finality that the planets in our solar system revolve about the sun.

How does Newton's Method work?

Let's see!

How does the process work?

Method 1

1. Pick an x-value that you believe is "close" to the actual zero.

2. Find the corresponding y-value and slope of the tangent line at that location.

3. Write the equation of the line.

4. Plug in zero for y because we are looking for where the graph will cross the x-axis

5. Solve for x. This is your new x-value and you will go through the entire process again.

Method 2

Newton-Raphson Short-Cut:

Pick an x-value that you believe is "close" to the actual zero and then use the following algorithm:

$$x_{n+1} = x_n - f(x_n)/f'(x_n)$$

Where did this come from???

Pt. Slope Form with Function Notation:

$$f(x_n) - f(x_{n+1}) = m(x_n - x_{n+1})$$

But we are dealing with tangent lines so the slope is representing the slope of the tangent line or $f'(x_n)$. So replacing this, we get:

$$f(x_n) - f(x_{n+1}) = f'(x_n)(x_n - x_{n+1})$$

Then, plug in zero for $f(x_{n+1})$ because we are looking for where the graph will cross the x-axis. So, now we have:

$$f(x_n) = f'(x_n)(x_n - x_{n+1})$$

Finally, solve for x_{n+1} we get:

$$x_{n+1} = x_n - f(x_n)/f'(x_n)$$

ex) Use Newton's Method to approximate the zeros of $f(x) = x^2 + x - 1$ until two successive approximations differ by less than 0.001.

Wait! If the directions say this in your book, just do 2 iterations and stop. $y = x^2 + x - 1$

tangent lines	$x_{n+1} = x_n - f(x_n)/f'(x_n)$
start with $x_1 = 1$	start with $x_1 = 1$
$f(1) = 1$	$f(1) = 1$
$f'(x) = 2x+1$	$f'(x) = 2x+1$
$f'(1) = 3$	$f'(1) = 3$
$y - 1 = 3(x-1)$ ← set $y = 0$	$x_2 = 1 - \frac{1}{3} = \frac{2}{3}$
$x_2 = \frac{-1}{3} + 1 = \frac{2}{3}$	$f\left(\frac{2}{3}\right) = \frac{4}{9} + \frac{2}{3} - 1 = \frac{1}{9}$
$f\left(\frac{2}{3}\right) = \frac{4}{9} + \frac{2}{3} - 1 = \frac{1}{9}$	$f'\left(\frac{2}{3}\right) = \frac{4}{3} + 1 = \frac{7}{3}$
$f'\left(\frac{2}{3}\right) = \frac{4}{3} + 1 = \frac{7}{3}$	$x_3 = \frac{2}{3} - \frac{\frac{1}{9}}{\frac{7}{3}}$
$y - \frac{1}{9} = \frac{7}{3}\left(x - \frac{2}{3}\right)$ ← set $= 0$	$= \frac{2}{3} - \frac{1}{9} \cdot \frac{3}{7} = \frac{2}{3} - \frac{1}{21} = \frac{13}{21}$
$x = -\frac{1}{9} \cdot \frac{3}{7} + \frac{2}{3} = -\frac{1}{21} + \frac{14}{21} = \frac{13}{21}$	

Special case of Newton's method: approximating the values of roots (this same method also works for a variety of other functions)

ex) Use Newton's method to approximate $\sqrt{7}$

Since the output of Newton's method are x-values (zeros) and not y-values, we need to rewrite the function so that the x-value will give us the output we are looking for:

If $x = \sqrt{7}$, then $x^2 = 7$ and $x^2 - 7 = 0$.

If we let $y = x^2 - 7$, then the zero would give us the $\sqrt{7}$.

tangent lines	$x_{n+1} = x_n - f(x_n)/f'(x_n)$
$x_1 = 3$ $\quad y' = 2x$	$x_1 = 3$ $\quad y' = 2x$
$y(3) = 2$	$y(3) = 2$
$y'(3) = 6$	$y'(3) = 6$
$y - 2 = 6(x - 3)$	$x_2 = 3 - \frac{2}{6} = \frac{8}{3}$
$x_2 = \frac{-2}{6} + 3 = \frac{8}{3}$	
$y\left(\frac{8}{3}\right) = \frac{64}{9} - 7 = \frac{1}{9}$	$y\left(\frac{8}{3}\right) = \frac{64}{9} - 7 = \frac{1}{9}$
$y'\left(\frac{8}{3}\right) = \frac{16}{3}$	$y'\left(\frac{8}{3}\right) = \frac{16}{3}$
$y - \frac{1}{9} = \frac{16}{3}\left(x - \frac{8}{3}\right)$	$x_3 = \frac{8}{3} - \frac{\frac{1}{9}}{\frac{16}{3}} = \frac{8}{3} - \frac{1}{\cancel{9}_{3}} \cdot \frac{\cancel{3}}{16}$
$x_3 = -\frac{1}{\cancel{9}_{3}} \cdot \frac{\cancel{3}}{16} + \frac{8}{3}$	$= \frac{128}{48} - \frac{1}{48} = \boxed{\frac{127}{48}}$
$x_3 = \frac{-1}{48} + \frac{128}{48} = \boxed{\frac{127}{48}}$	

How do Newton's method and linear approximation relate?

Linear approximation is basically just using the tangent line to approximate function values at points near the point of tangency. (Simply plug the x-value into the tangent line equation instead of the original function.)

Newton's method uses the tangent line to locate zeros of a function, and repeats the process to increase the accuracy.

ex) Find the tangent line approximation of $(3.1)^4$ using the point of tangency at x = 3.

$y = x^4$

$(3, 81)$

$y' = 4x^3$

$y'(1) = 108$

$y - 81 = 108(x - 3)$

$y - 81 = 108(3.1 - 3)$

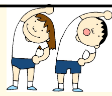
$y - 81 = 10.8$

$y = 91.8$

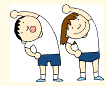
Newton's method does not always work!
AAARGGHH!!

Sometimes the zeros of the tangent lines do not converge. This can be for a variety of reasons.

(Change the GSP function to be $x^{(1/3)}$ and demonstrate using x_1 around 1/2.)



REVIEW!!



1. If $f(x)$ is continuous on [-3, 4] and differentiable on (-3, 4), which of the following must be true?
   a) there exists a value, c, in (-3, 4) such that $f'(c) = 0$
   b) there exists a value, c, in (-3, 4) such that $f'(c) = \frac{f(4) - f(-3)}{7}$
   c) there exists a value, c, in (-3, 4) such that $f(c) = 0$
   d) there exists a value, c, in (-3, 4) such that $f(c) = \frac{f(4) - f(-3)}{7}$
   e) None of these are necessarily true

2. If $f(x)$ is a continuous function with $f(6) = -2$, $f'(6) = 0$, and $f''(6) = 5$, which of the following is true?
   a) $f(x)$ has a relative maximum at (6, 0)
   b) $f(x)$ has a relative minimum at (6, 0)
   c) $f(x)$ has a relative maximum at (6, -2)
   d) $f(x)$ has a relative minimum at (6, -2)
   e) $f(x)$ has a point of inflection at (6, 5)

3. State the relative and absolute extrema for the function $f(x) = {}^1/_2\,x + \cos x$ on the interval $[0, 2\pi]$.

You try! Use Newton's Method to approximate the zeros of $f(x) = x^3 - \cos x$ until two successive approximations differ by less than 0.001.

(In other words, just do 2 iterations)

1) $f'(x) =$

2) Interval to focus on [a, b] is:

initial estimate, x1, is:

3)

What have we learned??

- Can I use Newton's method to approximate zeros?
- Can I use my calculator to help me with this?