15.8 Spherical Coordinates!!

Learning Target

- I can write a triple integral using spherical coordinates

Spherical coordinates

$x = \rho \sin\varphi \cos\theta$

$y = \rho \sin\varphi \sin\theta$

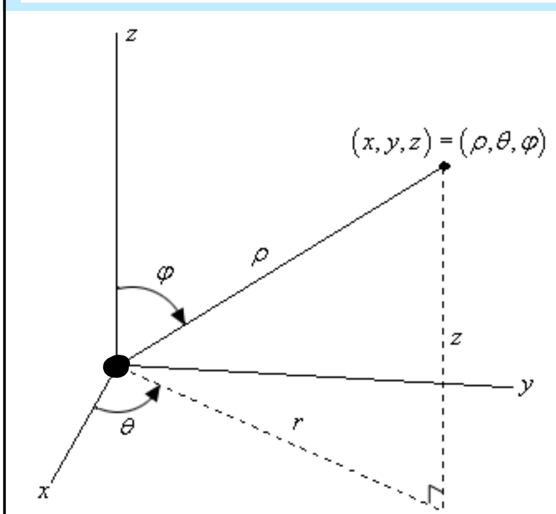
$z = \rho \cos\varphi$

$\rho^2 = x^2 + y^2 + z^2$

$r = \rho \sin\varphi$

Spherical coordinates are best used for shapes based on spheres and cones

$$\iiint_Q f(x,y,z)\,dz\,dy\,dx = \iiint_Q \underline{\rho^2 \sin\varphi} \cdot f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,d\rho\,d\varphi\,d\theta$$



Note: The limits for θ and φ are usually constants

To find the limits for φ, remember that $\cos\varphi = z/\rho$ and $\sin\varphi = r/\rho$

In spherical coordinates, the equation of a:

1) sphere of radius a centered at the origin is $\rho = a$

2) sphere of radius a centered at (0, 0, a) is $\rho = 2a\cos\varphi$

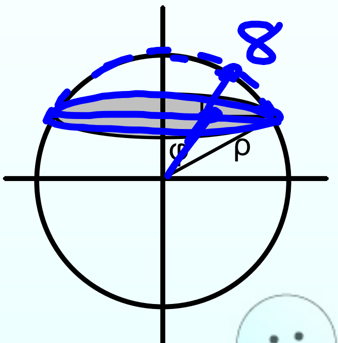
3) cone of angle a (from the z-axis) is $\varphi = a$

4) cylinder of radius a is $\rho = a/\sin\varphi$

5) horizontal plane at z = a: $\rho = a/\cos\varphi$

6) vertical plane that begins on the z-axis and goes out from there at an angle a (from the x-axis) is $\theta = a$

ex) Set up, but do not evaluate, 3 triple integrals (rectangular, cylindrical, and spherical) to find the volume of the solid below $x^2 + y^2 + z^2 = 64$ and above $z = 4$.

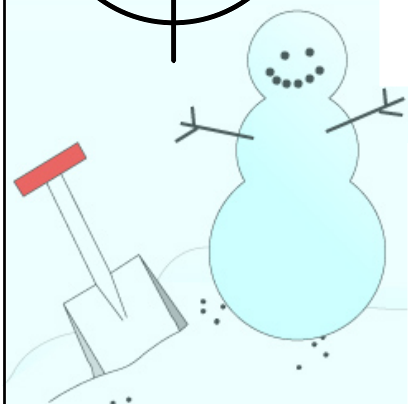


$x^2 + y^2 + 4^2 = 64 \rightarrow x^2 + y^2 = 48$ ✓

So the equation of the circle that is the 'base' of the region is $x^2 + y^2$

So our volume in rectangular coordinates would be:

$$V = \int_{-\sqrt{48}}^{\sqrt{48}} \int_{-\sqrt{48-x^2}}^{\sqrt{48-x^2}} \int_{4}^{\sqrt{64-x^2-y^2}} dz\, dy\, dx$$



Converting this to ~~polar~~ cylindrical coordinates we get:

$$V = \int_{0}^{2\pi} \int_{0}^{\sqrt{48}} \int_{4}^{\sqrt{64-r^2}} r\, dz\, dr\, d\theta$$

For spherical, we are no longer using z for the 'height' of the space. Our θ would still be from 0 to 2π.

Our φ would go from the y-axis $\varphi = 0$ to the angle where the base hits the sphere. This would be at the $\arctan \frac{\sqrt{48}}{4} = \arctan \sqrt{3} = \frac{\pi}{3}$.

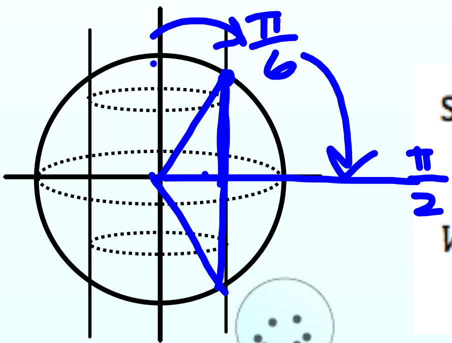
To convert from z to ρ, remember that $z = \rho \cos \varphi$, so $\rho = \frac{z}{\cos \varphi}$. Plugging in z = 4 we get $\rho = \frac{4}{\cos \varphi}$.

Plugging in a $z = \sqrt{64 - x^2 - y^2}$ and doing a lot of convoluted substituting and simplifying, we get $\rho = 8$.

So our volume in spherical coordinates would be:

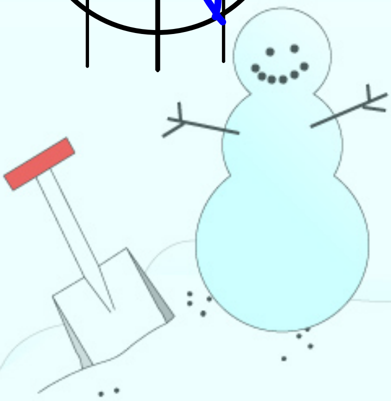
$$V = \int_{0}^{2\pi} \int_{0}^{\frac{\pi}{3}} \int_{\frac{4}{\cos\varphi}}^{8} \rho^2 \sin \varphi\, d\rho\, d\varphi\, d\theta$$

ex) Set up, but do not evaluate, 3 triple integrals (rectangular, cylindrical, and spherical) to find the volume of the solid inside of BOTH a sphere centered at the origin with a radius of 8 and a cylinder centered at the origin with a radius of 4.



So our volume in rectangular coordinates would be:

$$V = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{-\sqrt{64-x^2-y^2}}^{\sqrt{64-x^2-y^2}} dz\, dy\, dx$$



Converting this to cylindrical coordinates we get:

$$V = \int_{0}^{2\pi} \int_{0}^{4} \int_{-\sqrt{64-r^2}}^{\sqrt{64-r^2}} r\, dz\, dr\, d\theta$$

For spherical, there are several ways to go.

Method 1: Compute the integral that would give the volume of the ice cream cones and add the integral that would give the volume of the rest of the space in the cylinder.

$$V = 2\int_{0}^{2\pi} \int_{0}^{\frac{\pi}{6}} \int_{0}^{8} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta + 2\int_{0}^{2\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{0}^{\frac{4}{\sin\varphi}} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

Method 2: Compute the integral that would give the volume of the ice cream cones, then subtract the volume of the cones (leaving just the ice cream) and add the volume of the cylinder using geometry.

$$V = 2\int_{0}^{2\pi} \int_{0}^{\frac{\pi}{6}} \int_{0}^{8} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta - (2)\frac{1}{3}\pi(4^3) + 2\pi(4^2)\left(\sqrt{48}\right)$$

What have we learned?

- Can I write a triple integral using spherical coordinates?