Warmup (review) (No calculators!)

1) Find the average value of $f(x, y) = e^{x+y}$ over R where R is the triangle with vertices (0, 0), (0, 1), (1, 1)

2) Find the volume of the solid in the first octant bounded by $f(x, y) = 9 - x^2 - y^2$, $x = 0$, $y = 0$ and $y = 3 - x$

3) Find the center of mass of the lamina bounded by $x = 0$, $y = 0$ and $y = 2 - x$ with $\rho = 2x + y$

1)

$$Average\ Value = \frac{1}{\frac{1}{2}}\int_0^1\int_x^1 e^{x+y}\,dy\,dx$$

$$= 2\int_0^1 e^{x+y}\Big|_x^1\,dx = 2\int_0^1 (e^{x+1} - e^{2x})dx$$

$$= 2\left(e^{x+1} - \frac{1}{2}e^{2x}\right)\Big|_0^1$$

$$= 2\left(e^2 - \frac{1}{2}e^2\right) - 2\left(e - \frac{1}{2}\right) = e^2 - 2e + 1$$

2)

$$V = \int_0^3\int_0^{3-x} (9 - x^2 - y^2)\,dy\,dx$$

$$= \int_0^3 9y - x^2y - \frac{1}{3}y^3\Big|_0^{3-x}\,dx$$

$$= \int_0^3 \left(27 - 9x - 3x^2 + x^3 - \frac{1}{3}(3-x)^3\right)dx$$

$$= \int_0^3 \left(\frac{4}{3}x^3 - 6x^2 + 18\right)dx$$

$$= \frac{1}{3}x^4 - 2x^3 + 18x\Big|_0^3 = 27 - 54 + 54 = 27$$

3)

$$M_y = \int_0^2\int_0^{2-x} x(2x+y)\,dy\,dx = \int_0^2 2x^2y + \frac{1}{2}xy^2\Big|_0^{2-x}\,dx$$

$$= \int_0^2 \left(2x^2 - \frac{3}{2}x^3 + 2x\right)dx = \frac{2}{3}x^3 - \frac{3}{8}x^4 + x^2\Big|_0^2 = \frac{16}{3} - 6 + 4 = \frac{10}{3}$$

$$M_x = \int_0^2\int_0^{2-x} y(2x+y)\,dy\,dx = \int_0^2 xy^2 + \frac{1}{3}y^3\Big|_0^{2-x}\,dx$$

$$\int_0^2 \left(\frac{2}{3}x^3 - 2x^2 + \frac{8}{3}\right)dx = \frac{1}{6}x^4 - \frac{2}{3}x^3 + \frac{8}{3}x\Big|_0^2 = \frac{8}{3} - \frac{16}{3} + \frac{16}{3} = \frac{8}{3}$$

$$m = \int_0^2\int_0^{2-x} (2x+y)\,dy\,dx = \int_0^2 2xy + \frac{1}{2}y^2\Big|_0^{2-x}$$

$$\int_0^2 \left(-\frac{3}{2}x^2 + 2x + 2\right)dx = -\frac{1}{2}x^3 + x^2 + 2x\Big|_0^2 = -4 + 4 + 4 = 4$$

$$Center\ of\ mass = \left(\frac{\frac{10}{3}}{4}, \frac{\frac{8}{3}}{4}\right) = \left(\frac{5}{6}, \frac{2}{3}\right)$$

15.7 Cylindrical Coordinates!!

Learning Target

- I can write a triple integral using cylindrical coordinates

Believe it or not, it was because triple integrals were so difficult to evaluate using rectangular coordinates that alternate coordinate systems were created.

Cylindrical coordinates

(very similar to polar)

$x = r\cos\theta$

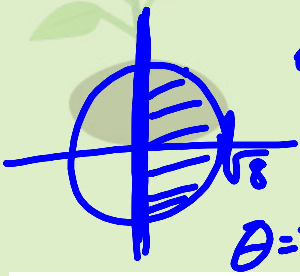
$y = r\sin\theta$

$z = z$

$$\iiint_Q f(x,y,z)\,dz\,dy\,dx = \iiint_Q r\cdot f(r\cos\theta, r\sin\theta, z)\,dz\,dr\,d\theta$$

Rewrite the following triple integral using cylindrical coordinates

$y^2 = 8 - x^2$



$\theta = \frac{\pi}{2}$, $r = 2\sqrt{2}$; $\theta = -\frac{\pi}{2}$, $r = 0$

$$\int_{0}^{2\sqrt{2}} \int_{-\sqrt{8-x^2}}^{\sqrt{8-x^2}} \int_{-1}^{2} \sqrt{1 + x^2 + y^2}\, dz\, dy\, dx$$

In cylindrical coordinates, z stays z, so we really just need to focus on converting the x and y-limits as well as the integrand.

The integrand is easy: $\sqrt{1 + x^2 + y^2} = \sqrt{1 + r^2}$ so that's done

Looking at the y's: $y = \sqrt{8 - x^2}$ is a semicircle, so the lower and upper limits together form the whole circle of radius $\sqrt{8}\ (or\ 2\sqrt{2})$ (just square both sides and rearrange the terms if you want to confirm this). So our r's would go from 0 to $2\sqrt{2}$ to create the same shape.

Since the x's go only from 0 to $2\sqrt{2}$, we are only talking about the right half of the circle. To get the same restriction with cylindricals, let θ go from $-\frac{\pi}{2}\ to\ \frac{\pi}{2}$.

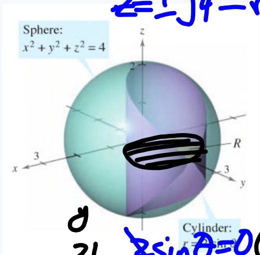
So our final integral would be:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2\sqrt{2}} \int_{-1}^{2} r\sqrt{1 + r^2}\, dz\, dr\, d\theta$$

Find the volume of the solid region cut from the sphere $(x^2 + y^2) + z^2 = 4$ by the cylinder $r = 2\sin\theta$

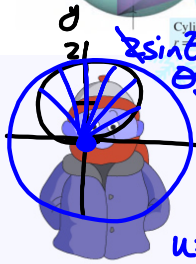
$z = \pm\sqrt{4 - r^2}$

(You may use a calculator to evaluate the integral)



$$V = \int_0^{\pi}\int_0^{2\sin\theta}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} r\,dz\,dr\,d\theta = 2\int_0^{\pi}\int_0^{2\sin\theta}\int_{0}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$$

$$= \frac{16}{3}\pi \text{ [redacted]}$$



$2\sin\theta = 0$

$\theta = 0, \pi$

$$2\int_0^{\pi}\int_0^{2\sin\theta} rz\Big|_0^{\sqrt{4-r^2}} dr\,d\theta$$

$$= 2\int_0^{\pi}\int_0^{2\sin\theta} r\sqrt{4-r^2}\,dr\,d\theta$$

$u = 4 - r^2$

$du = -2r\,dr$

$-du = dr$

$$= \int_0^{\pi}\int_{u_0}^{u_1} u^{\frac{1}{2}}\,du\,d\theta$$

$$= \int_0^{\pi} -\frac{2}{3}(4-r^2)^{\frac{3}{2}}\Big|_0^{2\sin\theta} d\theta$$

$$= \int_0^{\pi} -\frac{2}{3}\left[(4 - 4\sin^2\theta)^{\frac{3}{2}} - 8\right] d\theta$$

$$= -\frac{2}{3}\int_0^{\pi}\left[8(1-\sin^2\theta)^{\frac{3}{2}} - 8\right] d\theta$$

$$= -\frac{16}{3}\int_0^{\pi}(\cos^3\theta - 1)\,d\theta$$

$$= -\frac{16}{3}\int_0^{\pi}\left[\cos\theta(1-\sin^2\theta) - 1\right] d\theta$$

$$= -\frac{16}{3}\left[\int_0^{\pi}\cos\theta(1-\sin^2\theta)\,d\theta - \int_0^{\pi} 1\,d\theta\right]$$

$u = \sin\theta$

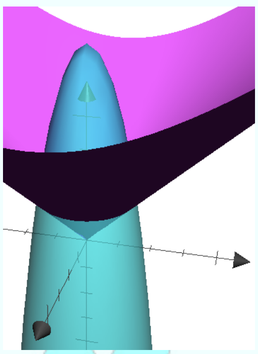
$du = \cos\theta\,d\theta$

$$= -\frac{16}{3}\left[\int (1-u^2)\,du \quad - \theta\Big|_0^{\pi}\right]$$

$$= -\frac{16}{3}\left[\sin\theta - \frac{1}{3}\sin^3\theta - \theta\Big|_0^{\pi}\right]$$

$$= -\frac{16}{3}\left[-\pi - 0\right] = \frac{16}{3}\pi$$

Find the volume of the solid between the cone and the inverted parabaloid with equations below.



cone: $z = \sqrt{x^2 + y^2}$ inverted parabaloid: $z = 12 - x^2 - y^2$

The set of all ordered pairs we need to use would be on the circle of intersection.

To find this, you can keep everything rectangular and then convert, or convert first.

$$\sqrt{r^2} = 12 - r^2$$

$$r^2 + r - 12 = 0$$

$$(r + 4)(r - 3) = 0$$

$r = 3$

$r = 3$ is the equation of the circle that makes sense because our z's are all positive.

So our volume would be:

$$\int_0^{2\pi} \int_0^{3} \int_r^{12-r^2} r \, dz \, dr \, d\theta$$

What have we learned?

- Can I write a triple integral using cylindrical or spherical coordinates?

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