Warmup (review) (calculators permitted)

1) ~~Find the volume of the solid above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$~~

2) ~~Find the volume of the solid inside the cylinder $x^2 + y^2 = 4$ and inside the ellipsoid $4x^2 + 4y^2 + z^2 = 64$~~

3) Find the volume under the surface $z = (x^2 + y^2)/y$ over R where R is the triangle with vertices (0, 0), (0, 4) and (4, 4)

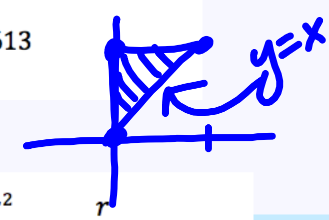
1)

$z = \sqrt{x^2+y^2} = r$ (note that z will always be positive)

$z^2 = 1 - x^2 - y^2 = 1 - r^2 \;\; so \;\; z = \sqrt{1-r^2}$

$r = \sqrt{1-r^2} \;\rightarrow\; r^2 = 1 - r^2 \;\rightarrow\; 2r^2 = 1 \;\rightarrow\; r = \frac{1}{\sqrt{2}}$

$$V = \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}} \left(\sqrt{1-r^2} - r\right) r\,dr\,d\theta \approx 0.613$$



2)

$x^2 + y^2 = 4 \;\; so \;\; r^2 = 4$

$z = \pm\sqrt{64 - 4x^2 - 4y^2} = \pm\sqrt{64 - 4r^2}$

$$V = 2\int_0^{2\pi}\int_0^{2} \sqrt{64-4r^2}\, r\,dr\,d\theta \approx 187.916$$

3)

$z = \frac{x^2+y^2}{y} = \frac{r^2}{r\sin\theta} = \frac{r}{\sin\theta}$

$$V = \int_0^4\int_x^4 \frac{(x^2+y^2)}{y}\,dy\,dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\frac{4}{\sin\theta}} \frac{r}{\sin\theta} r\,dr\,d\theta = \frac{256}{9} \approx 28.444$$

15.4 Center of Mass and Inertia!!

Learning Targets

- I can find the mass and center of mass of a lamina with variable density
- I can use double integrals to find the moment of inertia

What does this look like with 2-dimensional systems?

$$\bar{x} = \frac{10(1) + 2(5) + 5(-4)}{10 + 2 + 5} = 0$$

$$\bar{y} = \frac{10(-1) + 2(5) + 5(0)}{10 + 2 + 5} = 0$$

The center of mass is at the origin!

What the heck is a **planar lamina**?

A planar lamina is a thin, flat solid of some density, such as a sheet of paper or a sheet of metal. If the lamina is symmetrical about a point (such as a circle or rectangle), the center of mass would be at the center of the object. For a triangle, it would be at the centroid.

Think of it as though the object is rigid and you are trying to balance it on the head of a pin.

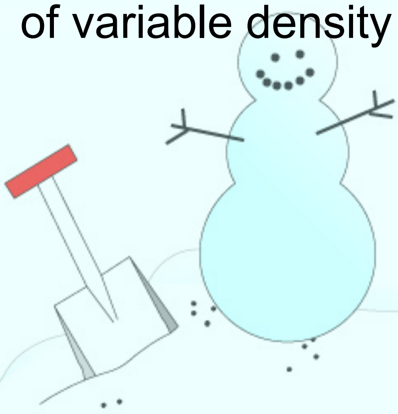
The mass of a planar lamina with a constant density is equal to its density x its area.

In chapter 7, students (not us, but some students out there somewhere) study how to find the mass and center of mass of a planar lamina of constant density. But we are calc 3 students, so we are going to learn how to calculate both the mass and the center of mass of a planar lamina of variable density!

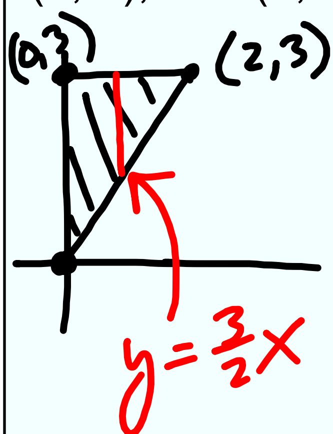
Density is represented by the greek letter ρ (rho), so our variable density will be a function, $\rho(x, y)$.

Since mass is just area x density, the mass of a planar lamina of variable density would be given by:

$$m = \iint_R \rho(x, y)\, dA$$



ex) Find the mass of the triangular lamina with vertices (0, 0), (0, 3), and (2, 3) given that the density is ρ(x, y) = 2x + y.



$$m = \int_0^2 \int_{\frac{3}{2}x}^{3} (2x + y)\, dy\, dx = \int_0^2 2xy + \frac{1}{2}y^2 \Big|_{\frac{3}{2}x}^{3} dx$$

$$= \int_0^2 \left(6x + \frac{9}{2} - 3x^2 - \frac{9}{8}x^2\right) dx = \int_0^2 6x + \frac{9}{2} - \frac{33}{8}x^2 dx$$

$$= 3x^2 + \frac{9}{2}x - \frac{11}{8}x^3 \Big|_0^2 = 12 + 9 - 11 = 10$$

What the heck is a moment?

Moments are sets of measures about some sort of central measurement. For example:

- Moments of mass
  - > zeroith moment = total mass
  - > first moment = center of mass
  - > second moment = rotational inertia
- Moments of statistics
  - > zeroith moment = total probability
  - > first moment = mean
  - > second moment = variance
  - > third moment = skewness
- Moments of force
  - > first moment = moment of inertia (torque)

Typically, the additional moments deal with powers of distances from the center. For example, for mass, the first moment is the integral of radius times density divided by total mass, the second moment is the integral of the square of the radius times density.

Don't confuse a moment with a moment in time. For example, when we say 'moment of inertia', it is not the moment in time at which inertia is exerted, it is a measure that calculates an object's resistence to rotation about an axis. It is based on the mass of the object and it's distance from the axis.

First Moments and Center of Mass

As is said on the previous screen, the center of mass is the first moment normalized by the total mass

NOTE: M_x is the moment about the x-axis (the vertical moment) and M_y is the moment about the y-axis (the horizontal moment)

The first moments of mass are:

$$M_y = \iint_R x\rho(x,y)\,dA \quad and \quad M_x = \iint_R y\rho(x,y)\,dA$$

The center of mass is:

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{m},\frac{M_x}{m}\right)$$

Note: the centroid is the geometric center of a planar lamina, while the center of mass takes density into account. If the density is uniform throughout the object, then the centroid and center of mass will be the same.

ex) Find the center of mass of the lamina corresponding to the parabolic region $0 \le y \le 4 - x^2$ where the density at any point (x, y) is proportional to the distance between (x, y) and the x-axis (so $\rho(x,y) = ky$).

Since our region and our density are both symmetric about the y-axis, it is a safe assumption that $M_y = 0$.

$$M_x = \int_{-2}^{2}\int_{0}^{4-x^2} ky^2\,dy\,dx = \int_{-2}^{2} \frac{1}{3}ky^3\Big|_0^{4-x^2} dx$$

$$= \int_{-2}^{2} \frac{1}{3}k(-x^6 + 12x^4 - 48x^2 + 64)dx$$

$$= \frac{1}{3}k\left(-\frac{1}{7}x^7 + \frac{12}{5}x^5 - 16x^3 + 64x\right)\Big|_{-2}^{2}$$

$$\frac{1}{3}k\left(-\frac{128}{7} + \frac{384}{5} - 128 + 128 - \left(\frac{128}{7} - \frac{384}{5} + 128 - 128\right)\right)$$

$$= \frac{1}{3}k\left(\frac{4096}{35}\right) = \frac{4096}{105}k$$

$$m = \int_{-2}^{2}\int_{0}^{4-x^2} ky\,dy\,dx$$

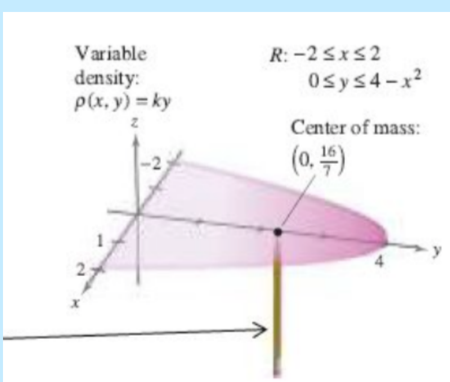
$$= \int_{-2}^{2} \frac{1}{2}ky^2\Big|_0^{4-x^2} dx$$

$$= \int_{-2}^{2} \frac{1}{2}k(16 - 8x^2 + x^4)dx$$

$$= \frac{1}{2}k\left(16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right)\Big|_{-2}^{2}$$

$$= \frac{1}{2}k\left(32 - \frac{64}{3} + \frac{32}{5} - \left(-32 + \frac{64}{3} - \frac{32}{5}\right)\right) = \frac{256}{15}k$$

$$\text{Center of mass} = \left(0, \frac{\frac{4096}{105}k}{\frac{256}{15}k}\right) = \left(0, \frac{16}{7}\right)$$



M_x and M_y are called the first moments about the x and y-axes. Remember that the first moment is the product of mass x distance. These moments represent the tendency of matter to resist a change in straight-line motion.

The **second moment** is called the **moment of inertia**. This is the measure of the tendency of matter to resist a change in rotational motion about a line and is the product of mass x distance^2.

$$I_y = \iint_R (x^2)\rho(x,y)dA \quad and \quad I_x = \iint_R (y^2)\rho(x,y)dA$$

ex) Write the double integral that would give the moment of inertia about the x-axis for the problem we just finished, $0 \leq y \leq 4 - x^2$ where $\rho = ky$.

✓ $$I_x = \int_{-2}^{2} \int_{0}^{4-x^2} ky^3 \, dy \, dx$$

What have we learned?

- Can I find the mass of a planar lamina of variable density?
- Can I find the center of mass of a planar lamina of variable density?
- Can I find the moment of inertia of a planar lamina of variable density?