Warmup!

Create a function, $f(x, y)$ where $f_x(x, y) = 2xy$

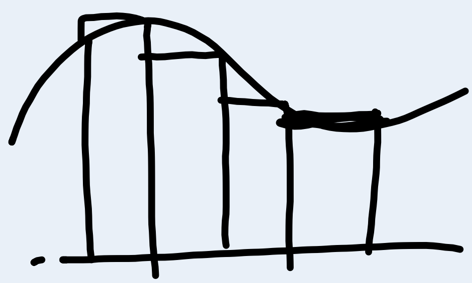
$$f(x,y) = x^2y + C$$

15.1a Volume and iterated integrals!!

Learning Targets

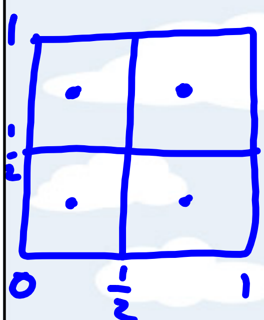
- I can use a double integral to represent the volume of a solid region
- I can evaluate a double integral as an iterated integral

How is volume calculated with double integrals?



So before we calculate exact volume using double integrals, let's learn how the integrals work by approximating the volume first!

ex) Use the midpoint rule to approximate the volume of the solid lying between the parabaloid below and the square region given by $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Use a partition made up of squares whose sides have a length of 1/2.



$$f(x,y) = 1 - \frac{1}{2}x^2 - \frac{1}{2}y^2$$

(The midpoint rule means that the heights of each prism are determined by the value of the function at the center of each square.)

The centers of the squares will be $\left(\frac{1}{4},\frac{1}{4}\right),\left(\frac{3}{4},\frac{1}{4}\right),\left(\frac{1}{4},\frac{3}{4}\right),\left(\frac{3}{4},\frac{3}{4}\right)$

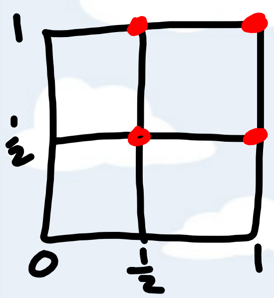
So the heights are: $f\left(\frac{1}{4},\frac{1}{4}\right) = \frac{30}{32} = \frac{15}{16}, f\left(\frac{3}{4},\frac{1}{4}\right) = \frac{22}{32} = \frac{11}{16}$

$$f\left(\frac{1}{4},\frac{3}{4}\right) = \frac{22}{32} = \frac{11}{16},\ f\left(\frac{3}{4},\frac{3}{4}\right) = \frac{14}{32} = \frac{7}{16}$$

Since the area of each square $= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$,

the approximate volume $= \frac{1}{4}\left(\frac{15}{16} + \frac{11}{16} + \frac{11}{16} + \frac{7}{16}\right) = \frac{1}{4}\left(\frac{44}{16}\right) = \frac{11}{16} \approx 0.6875$

ex) Ditto: this time using the upper right corner of each square to approximate the volume of the solid lying between the parabaloid below and the square region given by 0 ≤ x ≤ 1 and 0 ≤ y ≤ 1. Use a partition made up of squares whose sides have a length of 1/2.



$$f(x,y) = 1 - \frac{1}{2}x^2 - \frac{1}{2}y^2$$

The upper right corners of the squares will be $\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},1\right),\left(1,\frac{1}{2}\right),(1,1)$

So the heights are: $f\left(\frac{1}{2},\frac{1}{2}\right) = \frac{3}{4}, f\left(\frac{1}{2},1\right) = \frac{3}{8}$

$$f\left(1,\frac{1}{2}\right) = \frac{3}{8},\ f(1,1) = 0$$

Since the area of each square $= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$,

the approximate volume $= \frac{1}{4}\left(\frac{3}{4} + \frac{3}{8} + \frac{3}{8} + 0\right) = \frac{1}{4}\left(\frac{3}{2}\right) = \frac{3}{8} \approx 0.375$

Note, your book might refer to m and n when doing these problems. Just know that m = the number of intervals in the x-direction and n = the number of intervals in the y-direction.

For example, in the last 2 problems I gave you a side length of 1/2 for both directions, but I could have just said let m = 2 and n = 2 to accomplish the same thing. :)

$2 \le x \le 8, \quad 4 \le y \le 6$

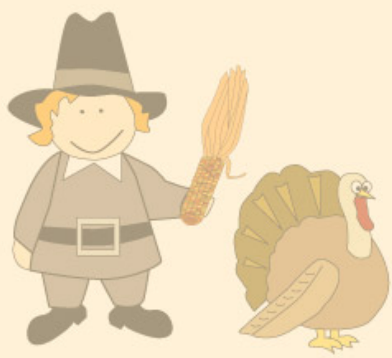
$m = 3, \quad n = 5 \qquad \text{width}_x = \frac{8-2}{3} = 2$

$\text{width} = \frac{b-a}{(m \text{ or } n)} \qquad \text{width}_y = \frac{6-4}{5} = \frac{2}{5}$

In chapter 14, we studied partial derivatives, which involved differentiating with respect to one variable while holding all other variables constant. In chapter 15, we are going to do the same thing with integrals.

Suppose we found the partial derivative $f_x(x, y) = 2xy$. This means that we held y constant while differentiating with respect to x. So if we wanted to 'undo' this operation, we would need to hold y constant again and integrate with respect to x.

$$\int 2xydx = y\int 2xdx = yx^2 + C(y)$$



Notice that the unknown constant does not have to be a 'constant', but can actually be a function in terms of y ex) $\int_{1}^{x}(2x^2y^{-2}+2y)dy =$

✓ $-\frac{2x^2}{y}+y^2\Big|_1^x = -2x + x^2 - (-2x^2+1) = 3x^2 - 2x - 1$

All right, you got this one, but can you take it up a notch?

$\int_{1}^{2}\int_{1}^{x}(2x^2y^{-2}+2y)dy\,dx$

✓ $= \int_{1}^{2}(3x^2-2x-1)dx = x^3 - x^2 - x|_1^2 = 8 - 4 - 2 - (1 - 1 - 1) = 3$

The double integral problem you just solved is called an 'iterated integral'. You work these from the inside out. The limits of the inside integral normally contain functions that are in terms of the outer variable, but the limits of the outside integral must be constants.

The limits identify the boundaries of the region of integration. Just like last year when you did an integral from 3 to 7 with respect to x, this bounded the region between x = 3 and x = 7, now with 2 variables the outer variable is bounded between constants but the inner variable can be bounded between 2 curves.

$$\int_{2}^{5}\int_{x+5}^{2x^2} f(x,y)\,dy\,dx$$

would describe a solid where the height is determined by f(x, y), and the base region is bounded between x = 2, x = 5, y = x + 5 and $y = 2x^2$

So what is a double integral?

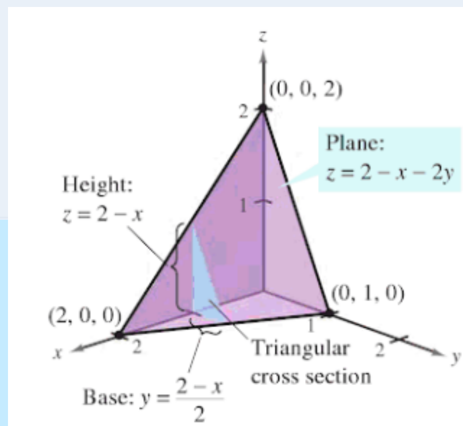
If f is defined on a closed, bounded region R in the xy-plane, then the double integral of f over R is:

$$\iint_R f(x,y)\,dA = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$$

The value of the double integral is equal to the volume of the region between f(x, y) and the xy-plane bounded by R

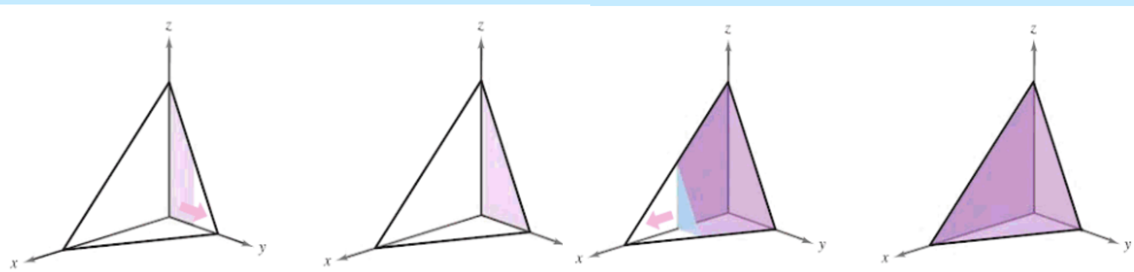
Using 'old' methods, we could find this volume by cross section:

$$\text{Volume: } \int_0^2 A(x)\,dx$$



Where A(x) is the area of the cross section.

Now we can use double integrals to find the same volume.



Integrate with respect to y to obtain the area of the cross section. Integrate with respect to x to obtain the volume of the solid.

$$\text{Volume} = \iint_R f(x,y)\,dA = \int_0^2 \int_0^{(2-x)/2} (2 - x - 2y)\,dy\,dx$$

ex) Evaluate $\iint_R \left(1-\frac{1}{2}x^2-\frac{1}{2}y^2\right)dA$

if 0 ≤ x ≤ 1, 0 ≤ y ≤ 1

$$\iint_R \left(1-\frac{1}{2}x^2-\frac{1}{2}y^2\right)dA = \int_0^1\int_0^1 \left(1-\frac{1}{2}x^2-\frac{1}{2}y^2\right)dy\,dx$$

$$\int_0^1 y-\frac{1}{2}x^2y-\frac{1}{6}y^3\Big|_0^1 dx = \int_0^1 \left(1-\frac{1}{2}x^2-\frac{1}{6}\right)dx$$

$$= x-\frac{1}{6}x^3-\frac{1}{6}x\Big|_0^1 = 1-\frac{1}{6}-\frac{1}{6}=\frac{2}{3}$$

This problem appears to be deceptively innocent. However it requires trig substitution which you did not learn last year. You might need it for some problems this year so let's walk through it together!

ex) Find the volume of the solid bounded by the paraboloid $z = 4 - x^2 - 2y^2$ and the xy-plane. 😈

1) Find the limits of your double integral

$$4 - x^2 = 0, \qquad x = \pm 2$$

$$4 - x^2 - 2y^2 = 0, \qquad y = \pm\sqrt{\frac{4-x^2}{2}}$$

2) Evaluate the 'dy' portion of the integral and simplify as much as possible

$$4\int_0^2 \int_0^{\sqrt{\frac{4-x^2}{2}}} (4 - x^2 - 2y^2)\,dy\,dx = 4\int_0^2 4y - x^2y - \frac{2}{3}y^3 \Big|_0^{\sqrt{\frac{4-x^2}{2}}} dx$$

$$= 4\int_0^2 \left(4\sqrt{\frac{4-x^2}{2}} - x^2\sqrt{\frac{4-x^2}{2}} - \frac{2}{3}\left(\frac{4-x^2}{2}\right)\sqrt{\frac{4-x^2}{2}}\right)dx$$

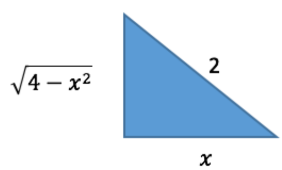
$$= 4\int_0^2 (4-x^2)\sqrt{\frac{4-x^2}{2}} - \frac{1}{3}(4-x^2)\sqrt{\frac{4-x^2}{2}}\,dx = 4\int_0^2 \frac{2}{3\sqrt{2}}(4-x^2)^{\frac{3}{2}}dx$$

$$= \frac{4\sqrt{2}}{3}\int_0^2 (4-x^2)^{\frac{3}{2}}dx$$

3) Set up your trig substitution

$$\sin\theta = \frac{\sqrt{4-x^2}}{2} \quad so\ 8\sin^3\theta = (4-x^2)^{\frac{3}{2}}$$

$$\cos\theta = \frac{x}{2} \quad so\ -2\sin\theta\,d\theta = dx$$



4) Rewrite your integral with trig

$$so\ \int_0^2 \frac{4\sqrt{2}}{3}(4-x^2)^{\frac{3}{2}}dx = \int_{\theta=\frac{\pi}{2}}^{\theta=0} -\frac{64\sqrt{2}}{3}\sin^4\theta\,d\theta = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{64\sqrt{2}}{3}\sin^4\theta\,d\theta$$

5) Apply some identities (pythagorean and power reduction will come in handy)

$$= \frac{64\sqrt{2}}{3}\int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin^2\theta\,(1-\cos^2\theta)d\theta = \frac{64\sqrt{2}}{3}\int_{\theta=0}^{\theta=\frac{\pi}{2}} \left(\frac{1}{2}\right)(1-\cos 2\theta)\left(1 - \left(\frac{1}{2}\right)(1+\cos 2\theta)\right)d\theta$$

$$= \frac{32\sqrt{2}}{3}\int_{\theta=0}^{\theta=\frac{\pi}{2}} (1-\cos 2\theta)\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)d\theta = \frac{16\sqrt{2}}{3}\int_{\theta=0}^{\theta=\frac{\pi}{2}} (1 - 2\cos 2\theta + \cos^2 2\theta)d\theta$$

6) Evaluate!

$$= \frac{16\sqrt{2}}{3}\int_{\theta=0}^{\theta=\frac{\pi}{2}} \left(1 - 2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta)\right)d\theta = \frac{16\sqrt{2}}{3}\int_{\theta=0}^{\theta=\frac{\pi}{2}} \left(\frac{3}{2} - 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right)d\theta$$

$$= \frac{16\sqrt{2}}{3}\left(\frac{3}{2}\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta\right)\Big|_{\theta=0}^{\theta=\frac{\pi}{2}} = \frac{16\sqrt{2}}{3}\left(\frac{3}{4}\pi - (0) + 0 - (0 - 0 + 0)\right) = 4\sqrt{2}\pi$$

Let's do some practice!

For each iterated integral below, write an equivalent integral changing the order of integration.

1) $\displaystyle\int_0^4 \int_0^{\sqrt{x}} dy\,dx = \int_0^2 \int_{y^2}^4 dx\,dy$ ✓

2) $\displaystyle\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} dx\,dy = \int_{-3}^3 \int_0^{\sqrt{9-x^2}} dy\,dx$

3) $\displaystyle\int_1^2 \int_0^{\ln x} dy\,dx = \int_0^{\ln 2} \int_{e^y}^2 dx\,dy$

4) $\displaystyle\int_0^1 \int_{4x}^4 dy\,dx = \int_0^4 \int_0^{\frac{1}{4}y} dx\,dy$

5) $\displaystyle\int_0^3 \int_0^{\sqrt{9-y}} dx\,dy = \int_0^{\sqrt{6}} \int_0^3 dy\,dx + \int_{\sqrt{6}}^3 \int_0^{9-x^2} dy\,dx$

6) $\displaystyle\int_0^1 \int_{\arctan x}^{\frac{\pi}{4}} dy\,dx = \int_0^{\frac{\pi}{4}} \int_0^{\tan y} dx\,dy$

What have we learned?

- Can I use a double integral to represent the volume of a solid region?
- Can I evaluate a double integral as an iterated integral?