14.8 Lagrange Multipliers!

Learning Target:

- I can use Lagrange multipliers to solve optimization problems involving functions of several variables

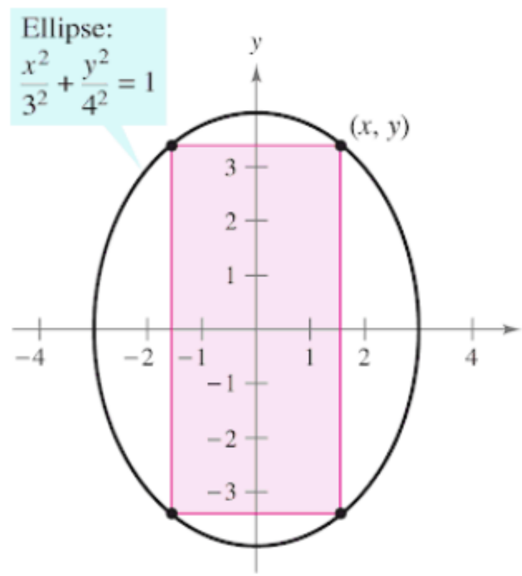
Background on Lagrange Multipliers

Suppose you want to find a rectangle of maximum area enclosed in the ellipse given by $x^2/9 + y^2/16 = 1$.

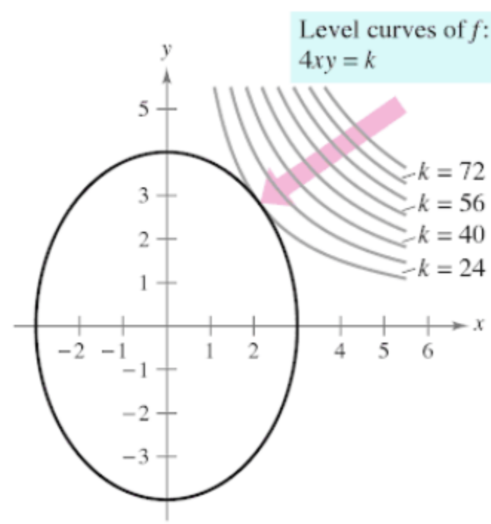
Since the ellipse is centered at the origin, if we call a point on the ellipse in the first quadrant (x, y), the area of the rectangle would be 2x * 2y. So f(x, y) = 4xy would be the equation we are trying to maximize. However, this is constrained by the ellipse function, $g(x, y) = x^2/9 + y^2/16 = 1$.

If we set f(x, y) = 4xy = k, we get the level curves for f. The level curves that are valid for the problem would be the ones that intersect the constraint function, and the one that produces the maximum for f would be the one that is tangent to g.

Remember, 2 curves are tangent if their gradient vectors are parallel. So one must be a scalar multiple of the other. We write this as $\nabla f(x, y) = \lambda \nabla g(x, y)$, where λ (lambda) is called the Lagrange multiplier.



Objective function: $f(x, y) = 4xy$



Constraint: $g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$

So how do we use Lagrange multipliers to solve max/min problems?

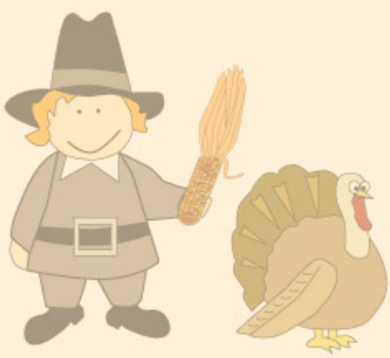
Suppose f(x, y) is the objective function and g(x, y) is the constraint where g(x, y) = c. To find the max/min of f:

1) Solve the system of equations:

$\nabla f(x, y) = \lambda \nabla g(x, y)$

$g(x, y) = c$

2) Test f(x, y) at each solution found in step 1. The largest value is the max, the smallest value is the min. :)



So back to our opener. If $f(x, y) = 4xy$ and $g(x, y) = x^2/9 + y^2/16 = 1$, then we get the following:

$\nabla f(x, y) = \lambda \nabla g(x, y)$ $\quad \langle 4y, 4x \rangle = \lambda \langle \frac{2}{9}x, \frac{1}{8}y \rangle$

$< 4y, 4x > = \lambda < 2x/9, 2y/16 >$

$4y = \lambda(2x/9)$

$4x = \lambda(2y/16)$

$x^2/9 + y^2/16 = 1$

$$y = \frac{2\lambda x}{36} = \frac{32x}{\lambda}$$

$$2\lambda^2 = 32 \cdot 36$$

$$\lambda^2 = 16 \cdot 36$$

$$\lambda = 24$$

$$\lambda = \frac{18y}{x} \quad so \quad 4x = \left(\frac{18y}{x}\right)\left(\frac{2y}{16}\right)$$

$$x^2 = \frac{9}{16}y^2$$

$$\frac{1}{16}y^2 + \frac{1}{16}y^2 = 1 \quad so \quad y^2 = 8 \quad and \quad y = \sqrt{8}$$

$$x^2 = \frac{9}{2} \quad so \quad x = \frac{3}{\sqrt{2}}$$

So the maximum value of $f = 4\left(\frac{3}{\sqrt{2}}\right)\left(\sqrt{8}\right) = 24$

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Use Lagrange multipliers to find the extrema:

$f(x, y) = x^2 - 10x + y^2 - 14y + 70$

x and y are both > 0 and $x + y = 10$

$f(x,y) = x^2 - 10x + y^2 - 14y + 70$ and $g(x,y) = x + y$

$2x - 10 = \lambda$

$2y - 14 = \lambda$

$x + y = 10$

$\lambda = 2x - 10 = 2y - 14$ so $y = x + 2$

$x + x + 2 = 10$ so $x = 4$ and $y = 6$

$f(x,y) = 16 - 40 + 36 - 84 + 70 = -2$

Test any other point that meets the constraints to determine if this is a max or min.

I tried $f(1,9) = 16 > -2$ so -2 is a minimum

Monday, we did the following problem. Try it again using Lagrange multipliers!

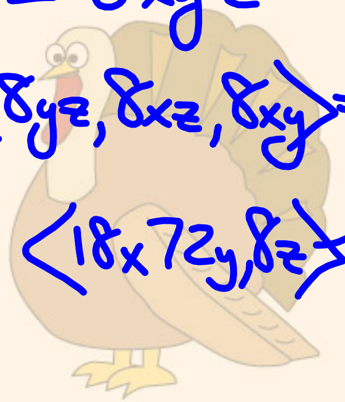
(Can you extend the method to 3 variables?)

Find the volume of the largest box that can fit in the ellipsoid:

$9x^2 + 36y^2 + 4z^2 = 36$

$V = 8xyz$

$\langle 8yz, 8xz, 8xy \rangle = \lambda \langle 18x, 72y, 8z \rangle$



$V = f(x, y, z) = 8xyz$

$g(x, y, z) = 9x^2 + 36y^2 + 4z^2$

So our system is:

$8yz = \lambda 18x$
$8xz = \lambda 72y$
$8xy = \lambda 8z$
$9x^2 + 36y^2 + 4z^2 = 36$

$$\lambda = \frac{4yz}{9x} = \frac{xz}{9y} = \frac{xy}{z}$$

So $y^2 = \frac{x^2}{4}$ and $z^2 = 9y^2$ so $z^2 = \frac{9x^2}{4}$

$$9x^2 + 36\left(\frac{x^2}{4}\right) + 4\left(\frac{9x^2}{4}\right) = 36$$

$$27x^2 = 36 \ \ so \ \ x^2 = \frac{4}{3} \ \ and \ \ x = \frac{2}{\sqrt{3}}$$

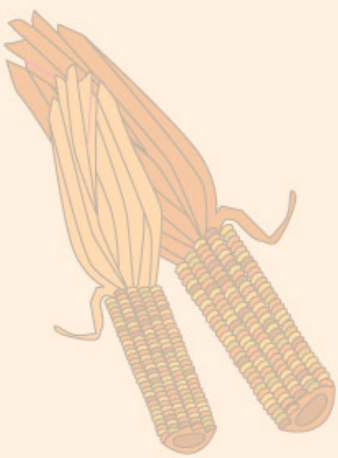
$$y = \frac{1}{\sqrt{3}} \ \ and \ z = \sqrt{3}$$

Max volume = $8\left(\frac{2}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)\left(\sqrt{3}\right) = \frac{16}{\sqrt{3}}$

What is there are multiple constraints?

You can introduce a second Lagrange multiplier (tradition is to use μ (mew)) to get:

$$\nabla f = \lambda \nabla g + \mu \nabla h$$



ex) Find the extrema for

$f(x, y, z) = x^2 + y^2 + z^2$ given that

x, y, z are ≥ 0, x + y = 12, x + 2z = 6

$f(x,y,z) = x^2 + y^2 + z^2$

$\langle 2x, 2y, 2z\rangle = \lambda\langle 1,1,0\rangle + \mu\langle 1,0,2\rangle$

$2x = \lambda + \mu$
$2y = \lambda$
$2z = 2\mu$
$x + y = 12$ and $x + 2z = 6$
Solving the system we get $x = 6, y = 6, z = 0$

$f(6,6,0) = 72$

Test $f(2,10,2) = 108 > 72$, so 72 is a minimum

What have we learned?

- Can I use Lagrange multipliers to solve constrained optimization problems?