Warmup!

Find the gradients of:

1) $f(x,y) = e^x \cos(4y)$

2) $g(x,y,z) = e^{2x}(\arctan y)(z^3)$

Find the directional derivatives of the following at the given point and in the direction of $\vec{v}$

3) $g(x,y) = 3x^2 + y - 5 \quad at \;\; P(1,3) \;\; with \; \vec{v} =< -2, 5 >$

4) $f(x,y) = \sin(2x)\cos y \quad at \;\; P\left(\frac{\pi}{4}, \frac{\pi}{4}\right) \;\; with \;\; \vec{v} =< 1, 2 >$

1) $\nabla f(x,y) = e^x \cos(4y)\,\boldsymbol{i} - 4e^x \sin(4y)\,\boldsymbol{j}$

2) $\nabla g(x,y,z) = \langle 2e^{2x}(\arctan y)(z^3), \frac{e^{2x}z^3}{1+y^2}, 3e^{2x}(\arctan y)(z^2)\rangle$

3) $\vec{u} = \langle -\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\rangle$

$$D_u f(x,y) = 6x\left(-\frac{2}{\sqrt{29}}\right) + (1)\left(\frac{5}{\sqrt{29}}\right)$$

$$D_u f(1,3) = -\frac{12}{\sqrt{29}} + \frac{5}{\sqrt{29}} = -\frac{7}{\sqrt{29}}$$

4) $\vec{u} = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\rangle$

$$D_u f(x,y) = 2\cos(2x)\cos y\left(\frac{1}{\sqrt{5}}\right) - \sin(2x)\sin y\left(\frac{2}{\sqrt{5}}\right)$$

$$D_u f(x,y) = \frac{2}{\sqrt{5}}(0) - \frac{2}{\sqrt{5}}(1)\left(\frac{\sqrt{2}}{2}\right) = -\sqrt{\frac{2}{5}}$$

14.6b Applications of Gradients

Learning Targets:

- I can use the gradient of a function of two variables in applications

ABCDEFGHIJKLMNOPQRSTUVWXYZ

Do you remember when we learned the formula for the angle between 2 vectors?

$$\cos\theta = \frac{u \cdot v}{||u||\,||v||}$$

If we rearrange this, we come up with an alternate definition for dot product:

$$u \cdot v = ||u||\,||v|| \cos\theta$$

(Which is why the dot product is zero implies that the vectors are orthogonal because the angle would have to be 90°.)

We've already established that the directional derivative is the dot product of the gradient and the unit vector, u, in a specific direction, so:

$$D_u(x,y) = \nabla f(x,y) \cdot u = ||\nabla f(x,y)||\,||u|| \cos\theta$$

But remember that u is a unit vector, so ||u|| = 1

$$D_u(x,y) = \nabla f(x,y) \cdot u = ||\nabla f(x,y)|| \cos\theta$$

What are the maximum and minimum values for cosθ? $-1 \le \cos\theta \le 1$

So the maximum value of the directional derivative occurs when cosθ = 1, which means that the **maximum value of the directional derivative (steepest slope of a surface at a specific point) = the magnitude of the gradient**

What else does this mean? If the steepest slope occurs when cosθ = 1, and if θ is the angle between the gradient and the vector u, then the steepest slope occurs when this angle is 0 which means the **steepest slope occurs in the direction of the gradient**.

Steepest Ascent

The direction of maximum increase of f = $\nabla f(x,y)$

The maximum value of $D_u f(x,y) = ||\nabla f(x,y)||$

Steepest Descent

The direction of minimum increase of f = $-\nabla f(x,y)$

The minimum value of $D_u f(x,y) = -||\nabla f(x,y)||$

ex) The temperature in °C on the surface of a metal plate is $T(x, y) = 20 - 4x^2 - y^2$. In what direction from (2, -3) does the temperature increase the most rapidly? What is the rate of increase?

$\nabla T(x, y) = \langle -8x, -2y \rangle$

$\nabla T(2, -3) = \langle -16, 6 \rangle$ which is the direction of maximum increase

$\|\nabla T(2, -3)\| = \sqrt{256 + 36} = \sqrt{292} \approx 17.09 \ ^\circ C/cm$

ex) The temperature in °C on the surface of a metal plate is $T(x, y) = 20 - 4x^2 - y^2$. If a heat-seeking particle is located at point (2, -3), Find the path of the particle.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Let $r(t) =< x(t), y(t) >$ be the path of the particle.

A vector tangent to each point on the path would be $r'(t) =< \frac{dx}{dt}, \frac{dy}{dt} >$.

Because the particle seeks maximum temperature, the directions of $r'(t)$ and $\nabla T(x,y) =< -8x, -2y >$ are the same at each point on the path.

So $\frac{dy}{dx} = \frac{-2y}{-8x} = \frac{y}{4x}$.

Separating the differential equation we get:

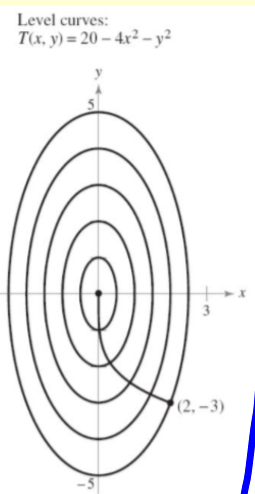
$$\int \frac{1}{y} dy = \int \frac{1}{4x} dx$$

$$\ln|y| = \frac{1}{4}\ln|x| + C$$

$$y = Cx^{\frac{1}{4}}$$

$$-3 = C * 2^{\frac{1}{4}} \text{ so } C = -\frac{3}{\sqrt[4]{2}} \approx -2.522$$

$$y = -2.522x^{\frac{1}{4}}$$



h followed by a heat-seeking particle

$$\ln|y| = \frac{1}{4}\ln|x| + C$$

$$|y| = e^{\frac{1}{4}\ln|x| + C}$$

$$|y| = e^{\ln|x|^{\frac{1}{4}}} \cdot e^{C}$$

$$|y| = A|x|^{\frac{1}{4}}$$

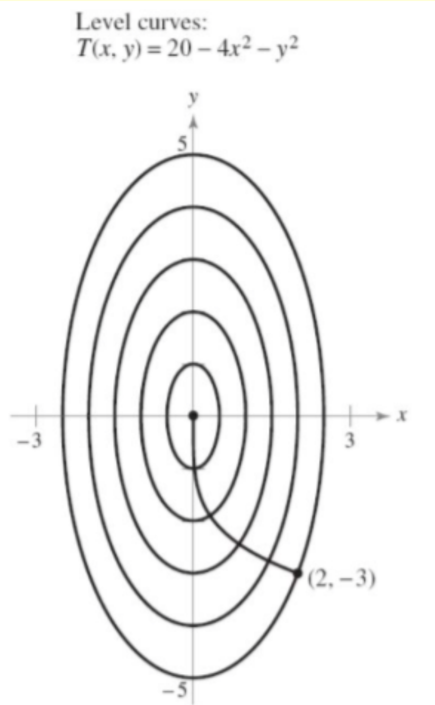
$$y = M|x|^{\frac{1}{4}}$$

$$-3 = M|2|^{\frac{1}{4}}$$

$$M = \frac{-3}{\sqrt[4]{2}}$$

$$y = \frac{-3}{\sqrt[4]{2}}|x|^{\frac{1}{4}}$$

One last note: If f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq 0$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0).



Path followed by a heat-seeking particle

Notice on the last example that the heat-seeking particle follows a path that is normal to each level curve.

ex) Find a vector normal to the level curve at point P if $f(x, y) = xy$, $c = -3$ and $P(-1, 3)$.

$\nabla f(x,y) = \langle y, x \rangle$

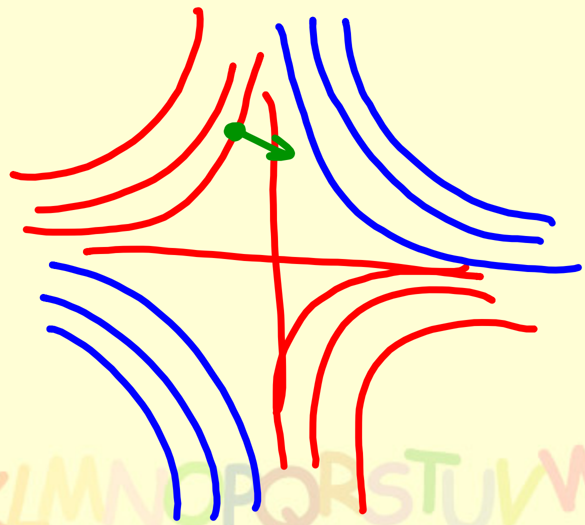
so the vector normal to the level curve at $P(-1, 3)$ is $\langle 3, -1 \rangle$

Note: the only way the level curve is actually used in this problem is to verify that the point on $f(x, y)$ is actually on the level curve

$xy = -3$

$y = -\frac{3}{x}$

$xy = -2$

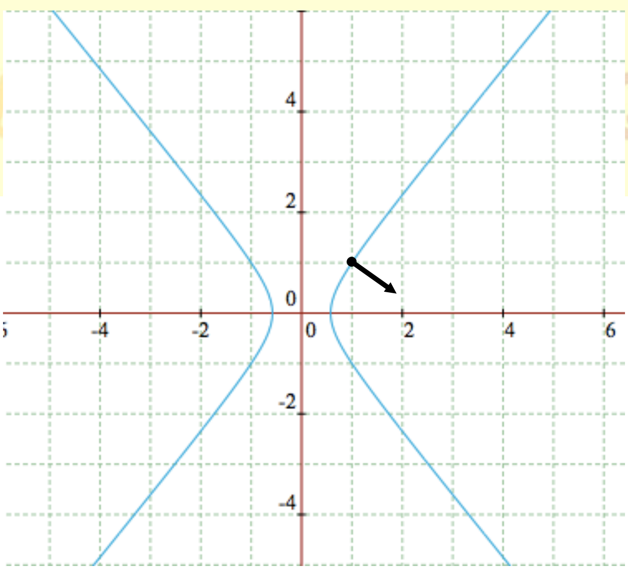


ex) Use the gradient to find a unit vector normal to the graph of $3x^2 - 2y^2 = 1$ at the point (1, 1).

$$\nabla f(x, y) = < 6x, -4y >$$

Normal vector through (1, 1) = <6, -4>

Unit normal vector = $\langle \frac{6}{\sqrt{52}}, -\frac{4}{\sqrt{52}} \rangle$ or $\langle \frac{3\sqrt{13}}{13}, -\frac{2\sqrt{13}}{13} \rangle$



What have we learned?

- Can I use the gradient of a function of two variables in applications?