Warmup: A cup in the shape of a cone is manufactured with the radius and height to be 3" and 7" respectively. If the errors of the radius and height are the same, what can the maximum error be so that the relative percent error of the volume is no more than 3.2%?

$$V = \frac{1}{3}\pi r^2 h$$

projected volume = $\frac{1}{3}\pi(9)(7) = 21\pi$

$\Delta r = \Delta h \; so \; I \; will \; use \; \Delta r \; for \; both$

$$dV = \frac{2}{3}\pi r h \Delta r + \frac{1}{3}\pi r^2 \Delta r$$

$$= \frac{2}{3}\pi(3)(7)\Delta r + \frac{1}{3}\pi(9)\Delta r = 14\pi\Delta r + 3\pi\Delta r = 17\pi\Delta r$$

relative percent error = $\frac{17\pi\Delta r}{21\pi} \leq 0.032$

$\Delta r \leq 0.0395$"

14.5 Chain Rule with Multiple Variables

Learning Targets:

- I can apply the chain rule to functions of several variables
- I can find partial derivatives implicitly

Warmup #2

Suppose w = $x^2y - y^2$ where x = sint and y = e^t.
Find dw/dt when t = 0.

Method 1

$$w = \sin^2 t \cdot e^t - e^{2t}$$

$$\frac{dw}{dt} = 2\sin t\cos t \cdot e^t + \sin^2 t \cdot e^t - 2e^{2t}$$

$$\left.\frac{dw}{dt}\right|_{t=0} = -2e^0 = -2$$

Method 2

$$\frac{dw}{dt} = 2xy\frac{dx}{dt} + x^2\frac{dy}{dt} - 2y\frac{dy}{dt}$$

$$\frac{dw}{dt} = 2\sin t \cdot e^t \cdot \cos t + \sin^2 t \cdot e^t - 2e^t \cdot e^t$$

Old Chain Rule:

When we spoke of chain rule last year, we learned it as: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

However, there is an alternate way of writing it which is: $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

'outer derivative' 'inner derivative'

We also learned last week that: $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

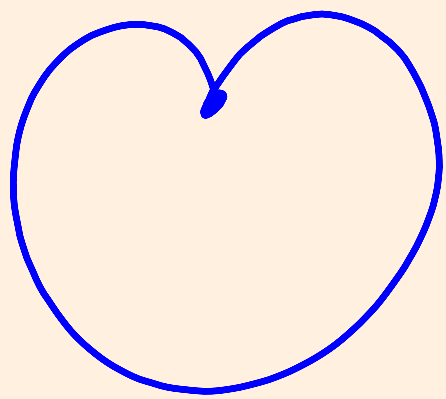
If we put these two concepts together we get a new chain rule for a multi-variable function whose inputs are actually functions in terms of a single variable (such as t)

New Chain Rule: $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

(which can be extended to more variables following the same pattern)

What does this mean?

If your x and y are parametrically defined in terms of t, then these equations generate a curve where the input is t and the output is (x, y). The set of points defined by this curve would then be input values for a z-function and dz/dt would be the rate of change of z as the t-value changes (meaning the rate of change of z as you move along the curve).



ex) The pressure P (in kilopascals), volume V (in liters) and temperature T (in kelvins) of a mole of an ideal gas are related by the equation PV = 8.31T. Find the rate at which the pressure is changing when the temperature is 300K and is increasing at a rate of 0.1 K/s and the volume is 100L and is increasing at a rate of 0.2 L/s.

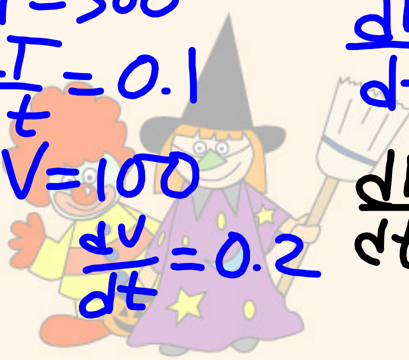
$\frac{dP}{dt} = ?$

$T = 300$

$\frac{dT}{dt} = 0.1$

$V = 100$

$\frac{dV}{dt} = 0.2$



$$P = \frac{8.31T}{V}$$

$$\frac{dP}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt}$$

$$\frac{dP}{dt} = \frac{8.31}{V}\left(\frac{dT}{dt}\right) + \frac{-8.31T}{V^2}\left(\frac{dV}{dt}\right)$$

$$\approx .00831 - .04986$$

$$\approx -.04155 \ \frac{KPa}{s}$$

What if the input functions are in terms of more than one variable?

Suppose z = f(x, y), x = g(s, t) and y = h(s, t). Then:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ for $w = 2xy$ where $x = s^2 + t^2$ and $y = \frac{s}{t}$.

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s}$$

1) Do this by substitution

2) Do this again using chain rule

1)

$$w = 2(s^2 + t^2)\left(\frac{s}{t}\right) = \frac{2(s^3 + st^2)}{t}$$

$$\frac{\partial w}{\partial s} = \frac{2(3s^2 + t^2)}{t} = \frac{6s^2 + 2t^2}{t}$$

$$\frac{\partial w}{\partial t} = \frac{2(s^3 + 2st)t - 2(s^3 + st^2)}{t^2} = \frac{2s^3 t - 2s^3}{t^2}$$

2)

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} = (2y)(2s) + (2x)\left(\frac{1}{t}\right)$$

$$= \left(\frac{2s}{t}\right)(2s) + 2(s^2 + t^2)\left(\frac{1}{t}\right) = \frac{4s^2 + 2s^2 + t^2}{t} = \frac{6s^2 + t^2}{t}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t} = (2y)(2t) + (2x)\left(-\frac{s}{t^2}\right)$$

$$= \left(\frac{2s}{t}\right)(2t) + 2(s^2 + t^2)\left(-\frac{s}{t^2}\right) = \frac{2st^2 - 2s^3}{t^2}$$

Ex) If $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s\sin t$, find the value of $\frac{\partial u}{\partial s}$ when $r = 2, s = 1, t = 0$.

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$= (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2\sin t)$$

$x|_{r=2,s=1,t=0} = 2,\quad y|_{r=2,s=1,t=0} = 2,\quad z|_{r=2,s=1,t=0} = 0$

$$\left.\frac{\partial u}{\partial s}\right|_{r=2,s=1,t=0} = (64)(2) + (16)(4) + (0)(0) = 192$$

What have we learned?

- Can I apply the chain rule to functions of several variables?
- Can I find partial derivatives implicitly?