Warmup: The 'Fall to Pieces' quilting company produces fabric squares to use for patchwork quilts. The squares are supposed to have a side length of 4", and the company promises that the length will never be 'off' by more than 0.01". What is the worst possible scenario for error in the area of their squares?

$(4.01)^2 = 16.0801$

✓ Since the expected area would be 16, the worst possible error in the area would be 0.0801 square inches.

14.4b Differentials

Learning Targets:

- I understand the concepts of increments and differentials
- I can use a differential as an approximation

What did we skip last year?

The next 9 slides are actually from section 3.9, which we covered only very briefly last year (you're welcome)

Write the equation of the line tangent to f(x) at (c, f(c)). ✓ $y - f(c) = f'(c)(x - c)$

$x - c$ is denoted as Δx

and $y - f(c)$ is denoted as Δy.

So we can rewrite the equation as:

$$\Delta y = f'(c)\Delta x$$

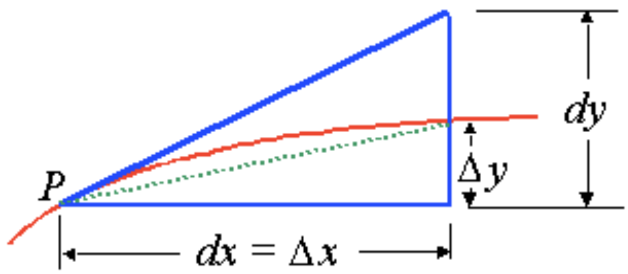
When Δx is small (approaching zero), we call Δx, dx, which is called the 'differential' of x.

Similarly, as Δx approaches zero, Δy becomes known as dy, the differential of y.

So now, $\Delta y = f'(c)\Delta x$

becomes $dy = f'(x)\,dx$

(think $dy = dy/dx * dx$)



Remember that $\frac{\Delta y}{\Delta x}$ represents slope of a secant line that connects 2 points on a function.

dy/dx represents slope of a line tangent to a function at a specific point.

To compare Δy and dy, think like this:

$$dy = f'(x)\,dx$$

$$\Delta y = f(x + \Delta x) - f(x)$$

ex) Compare Δy and dy for $y = 3x^2 + 2$ at $x = 1$ if $\Delta x = dx = 0.1$.

2nd Warmup: We have done so well with our squares, that we've moved on to circles! The radius of our circles is 6" and with an estimated error of no greater than 0.02". What is the worst possible scenario for error in the area of our circles?

3rd Warmup: Wow, the squares and circles are doing great! So great that now we're making spheres! The spheres have a radius of 5" which will never actually be off by more than .03". What is the worst possible scenario for error in the volume of our spheres?

Error Propagation!!

Differentials are often used in the approximation of error in measuring devices. In order to do this, let

x = measured value

$x + \Delta x$ = exact value

So Δx = error in measurement

If x is used to compute another value, than the difference between f(x) and f(x +Δx) is called the **propagated error**.

$$\text{Prop. error} = \Delta y = f(x + \Delta x) - f(x)$$

Summary of error formulas:

propagated error = $\triangle y = f(x + \triangle x) - f(x)$

estimated propagated error = $dy = f\,'(x)dx$

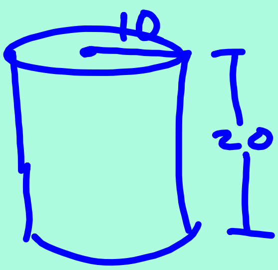
% error = $dy / f(x)$

A company produces cylinders whose heights are twice their radii. If the radius of the produced cylinder is 10 inches with an error of 0.25 inches:

a) What is the propagated error of the volume?

b) What is the estimated propagated error of the volume?

c) Using the answer from b), what is the percent error of the volume?



$V = \pi r^2 h \qquad \approx 6283.19$

$V = \pi(10)^2(20) = 2000\pi \text{ in}^3$

$V = \pi(10.25)^2(20.5) = \frac{68921}{32}\pi \text{ in}^3 \approx 6766.3$

prop. error $= 6766.3 - 6283.19$

$\approx 483.118 \text{ in}^3$

b) $V = \pi r^2 h$

$V = \pi r^2(2r)$

$V = 2\pi r^3$

$\frac{dV}{dr} = 6\pi r^2$

$dV = 6\pi r^2 dr$

$= 6\pi(10)^2(.25) \approx 471.24 \text{ in}^3$

The Big Fuzzy Dice Company is producing dice in the shapes of cubes with side lengths of 7". They have estimated an error of around 0.25" in the lengths of the sides of their cubes.

a) Find the propagated error, estimated propagated error, and percent error for the volume of each die

b) Find the propagated error, estimated propagated error, and percent error for the surface area of each die

For really small values of Δx and Δy, you can use the approximation $\Delta z \approx dz$.

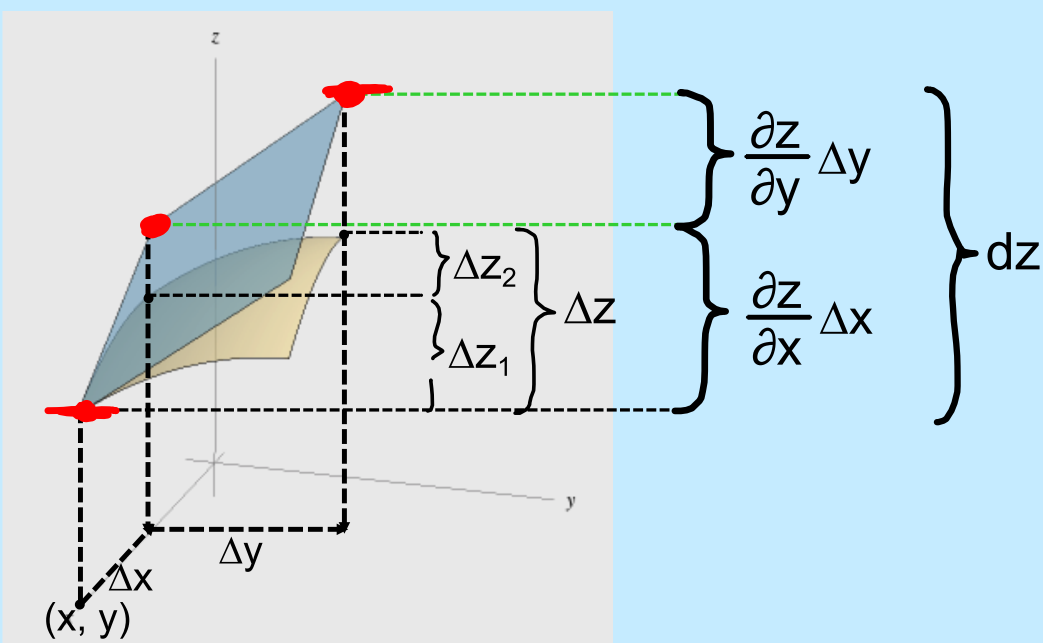
Great video, just watch the first couple minutes

Putting it together:

$\partial z/\partial x$ = the slope of the surface, z, in the x-direction

$\partial z/\partial y$ = the slope of the surface, z, in the y-direction

So, $dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$ represents the change in height of a plane that is tangent to the surface at the point (x, y, f(x,y)).



Use the differential dz to approximate the change in $z = \sqrt{4 - x^2 - y^2}$ as (x, y) moves from the point (1, 1) to the point (1.01, 0.97). Compare this approximation with the exact change in z, Δz.

$$\Delta z = \sqrt{4-(1.01)^2-(0.97)^2} - \sqrt{4-1^2-1^2} \approx 0.01372$$

$$\Delta x = 0.01 \quad \Delta y = -0.03$$

$$\frac{\partial z}{\partial x} = \frac{-2x}{2\sqrt{4-x^2-y^2}} = \frac{-x}{\sqrt{4-x^2-y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{-2y}{2\sqrt{4-x^2-y^2}} = \frac{-y}{\sqrt{4-x^2-y^2}}$$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

(1,1) (1,1) Δx Δy .01 -.03

$$\left.\frac{\partial z}{\partial x}\Delta x\right|_{x=1,y=1,\Delta x=0.01} = \frac{-1}{\sqrt{2}}(0.01) \approx -0.007071$$

$$\left.\frac{\partial z}{\partial y}\Delta y\right|_{x=1,y=1,\Delta x=0.01} = \frac{-1}{\sqrt{2}}(-0.03) \approx 0.021213$$

$$dz \approx -0.007071 + 0.021213 \approx \boxed{0.01414}$$

The possible error involved in measuring each dimension of a rectangular box is ±0.1 mm. The dimensions of the box are x = 50 cm, y = 20 cm and z = 15 cm. Use dV to estimate the propagated error and the relative percent error in the calculated volume of the box.

$V = xyz$

projected volume = (50)(20)(15) = 15000 cm^3

$dV = yz\Delta x + xz\Delta y + xy\Delta z$

.1mm = .01cm

estimated propagated error =

$$dV = (20)(15)(.01) + (50)(15)(.01) + (50)(20)(.01)$$

$$= 20.5\ cm^3$$

relative error $\approx \frac{20.5}{15000} \approx 0.0013667 \approx$ .1367%

What have we learned?

- Do I understand the concepts of increments and differentials?
- Can I use a differential as an approximation?