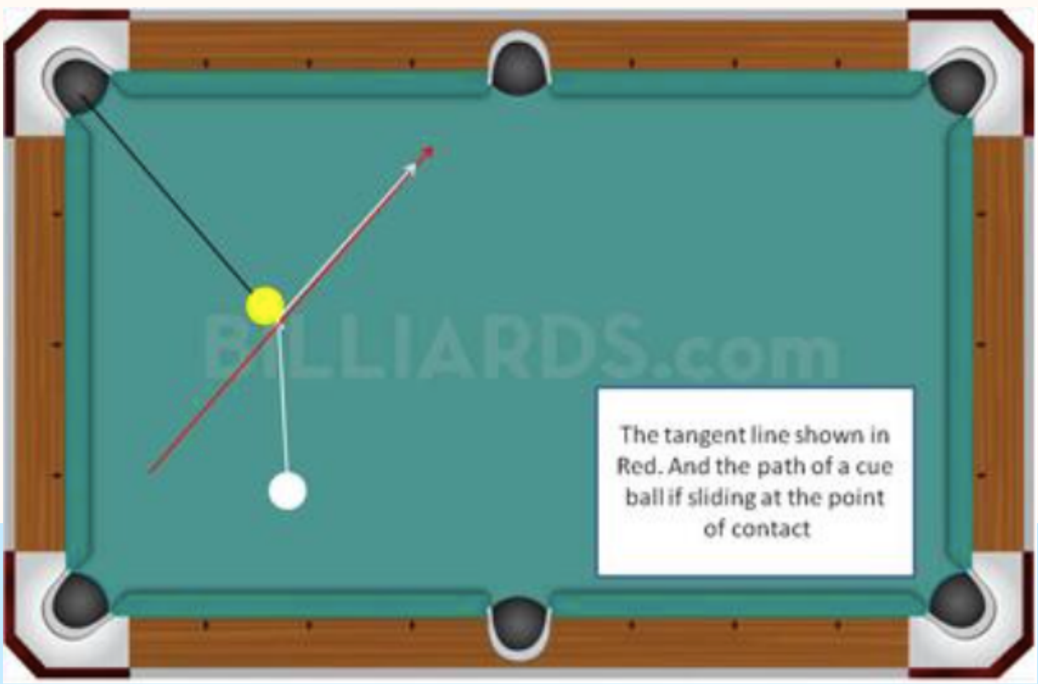
14.4a Tangent Planes and Differentials

Learning Target:

- I can find the equation of a plane tangent to a surface
- I can find the differential of a function

Let's Play Pool!!

The yellow ball will ALWAYS move in the direction normal to the point of impact, regardless of the direction of the cue ball.



So far, the majority of our equations in 3-space have been in the form of $z = f(x, y)$. For this lesson, we are going to start writing these a bit differently.

If $z = f(x, y)$, then let $F(x, y, z) = f(x, y) - z$

Since $z = f(x, y)$, then $f(x, y) - z = 0$ so $F(x, y, z) = 0$. Think of the surface described this way as being the level surface of F given by $F(x, y, z) = 0$.

Note that when we are finding the level surface where $f(x, y) = \#$, this produces a plane because z is a constant. However, once we get to $f(x, y, z) = \#$, the level surface is not necessarily 'flat', it simply implies a type of equation where a function of 3 inputs is equal to a constant.

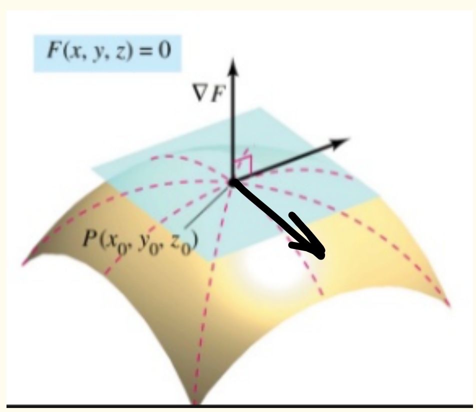
For example, if $F(x, y, z) = x^2 + y^2 + z^2 - 4$, then $F(x, y, z) = 0$ would be the level surface $x^2 + y^2 + z^2 = 4$ which is a sphere of radius 2.

Tangent Planes and Normal Lines

Let S be a surface given by F(x, y, z) = 0 and let $P(x_0, y_0, z_0)$ be a point on S.

1) The plane through P that contains 2 lines tangent to S at P is called the plane tangent to S at P. The equation of this plane is given by:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$



$F(x,y,z) = z^2 - 2x^2 - 2y^2 - 12$

ex) Find an equation of the plane tangent to the hyperboloid given by $z^2 - 2x^2 - 2y^2 = 12$ at the point (1, -1, 4).

Then find the equation of the line normal to the hyperboloid at the same point.

$F_x = -4x \quad F_x(1,-1,4) = -4$

$F_y = -4y \quad F_y(1,-1,4) = 4$

$F_z = 2z \quad F_z(1,-1,4) = 8$

$z^2 - 2x^2 - 2y^2 = 12$

so $F(x, y, z) = z^2 - 2x^2 - 2y^2 - 12$

$\nabla F(x, y, z) = \langle -4x, -4y, 2z \rangle$

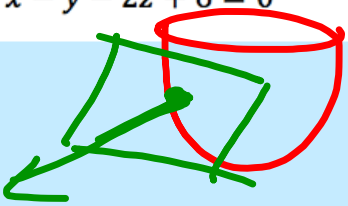
$\nabla F(1, -1, 4) = \langle -4, 4, 8 \rangle$

so the equation of the tangent plane is:

$-4(x-1) + 4(y+1) + 8(z-4) = 0$

or $(x-1) - (y+1) - 2(z-4) = 0$

or $x - y - 2z + 6 = 0$



line

$x = -4t + 1$

$y = 4t - 1$

$z = 8t + 4$

and the symmetric equations of the normal line are:

$$\frac{x-1}{-4} = \frac{y+1}{4} = \frac{z-4}{8}$$

You try!

Find the equation of the plane tangent to the surface x = y(2z - 3) at the point (4, 4, 2). Then find the symmetric equations of the line normal to the surface at the same point.

$F(x, y, z) = 2yz - 3y - x$

$\nabla F(x, y, z) = \langle -1, 2z - 3, 2y \rangle$

$\nabla F(4, 4, 2) = \langle -1, 1, 8 \rangle$

so the equation of the tangent plane is:

$-(x - 4) + (y - 4) + 8(z - 2) = 0$

or $x - y - 8z + 16 = 0$

and the symmetric equation of the normal line are:

$$-(x - 4) = y - 4 = \frac{z - 2}{8}$$

NOTE: Your new book solves all equations for z before asking you for tangent planes and doesn't expect you to convert the equations to 'surface' form before solving. So the partial z values for the problems tonight will always be 1. Let's do one in the spirit of what you'll see tonight just to make sure we've got it. Find the equation of the plan tangent to the curve of intersection of $z = x^2 - 2xy + y^2 - 5$ at the point (1, 2, -4)

$$z_x = 2x - 2y \qquad z_x(1,2,-4) = -2$$

$$z_y = -2x + 2y \qquad z_y(1,2,-4) = 2$$

$$z + 4 = -2(x-1) + 2(y-2)$$

Differentials!

If $z = f(x, y)$, then the TOTAL DIFFERENTIAL of z is:

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = f_x(x, y)dx + f_y(x, y)dy$$

This can be expanded to functions with 3 or more variables just by following the same pattern.

ex) Find the differential for $z = 2x\sin y - 3x^2y^2$

$$z_x = 2\sin y - 6xy^2$$

$$z_y = 2x\cos y - 6x^2y$$

$$dz = (2\sin y - 6xy^2)dx + (2x\cos y - 6x^2y)dy$$

What have we learned?

- Can I find the equation of a plane tangent to a surface?
- Can I find the equation of a line normal to a surface?