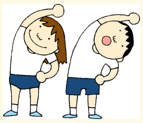
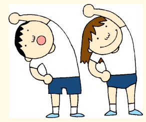
 WARM UP!! 

1) Give the values for which $r(t) = \langle e^t - \sin t, t^3 - 1, \tan t - t \rangle$ is NOT smooth

2) Write a vector valued function for a particle whose path passes through the points (5, 3, 2) and (-1, 8, 4).

1) $r'(t) = \langle e^t - \cos t, 3t^2, \sec^2 t - 1 \rangle$

The x and y components are never undefined, however the z component is undefined whenever cost = 0 which would be at π/2, and all of it's multiples, so we could write this as t = π/2 + kπ, k is an integer.

For r'(t) = < 0, 0, 0 >, the only t-value we need to check is at 0 because of the y-component. At t = 0, x(0) = 0, y(0) = 0 and z(0) = 0 so this would also be a place where the curve is not smooth.

2) Choosing random values for t of ~~r(0) = < 5, 3, 2 >~~ and ~~r(1) = < -1, 8, 4 >~~, I found ~~r(t) = < -6t + 5, 5t + 3, 2t + 2 >~~

t	x	y	z
0	5	3	2
1	-1	8	4

$$\boxed{\vec{u} = \langle -6, 5, 2 \rangle \ \therefore \ x = -6t + 5,\ y = 5t + 3,\ z = 2t + 2}$$

$\cancel{m = \frac{-6}{1}}$

$\cancel{x - 5 = -6(t - 0)}$

13.3b Tangent and Normal Vectors!

LEARNING TARGETS
At the end of this lesson, you will be able to:

- find a unit tangent vector and principal unit normal vector at a point on a space curve
- find the binormal vector at a point on a space curve
- find teh equatinos of the normal plane and osculating plane at a point on a space curve

If r(t) is a smooth curve, then the **unit tangent vector**, T(t) is:

$$T(t) = \frac{r'(t)}{||r'(t)||}$$

The unit tangent vector is a vector whose direction is the same as the direction of the curve at any given value of t

ex) Find the unit vector, T(t), tangent to the curve r(t) = <t, t^2>. Then find T(1).

r'

✓ $T(t) = \dfrac{\langle 1, 2t\rangle}{\sqrt{1+4t^2}}$ $\quad$ $so\ T(1) = \dfrac{\langle 1, 2\rangle}{\sqrt{5}}$

||r'||

A line tangent to a curve at a point is the line passing through the point that is parallel to the unit tangent vector.

ex) If r(t) = < 2cost, 2sint, t >, find T(t). Then find a set of parametric equations for the line tangent to the helix given by r(t) at t = π/4.

$r' = \langle -2\sin t, 2\cos t, 1\rangle$

$\|r'\| = \sqrt{4\sin^2 t + 4\cos^2 t + 1} = \sqrt{5}$

$$T(t) = \frac{\langle -2\sin t, 2\cos t, 1\rangle}{\sqrt{5}}$$

$$\boxed{T\left(\frac{\pi}{4}\right) = \frac{\langle -\sqrt{2}, \sqrt{2}, 1\rangle}{\sqrt{5}}}$$

$\sin^2(2t) + \cos^2(2t) = 1$

so the direction vector could be: $\boxed{\langle -\sqrt{2}, \sqrt{2}, 1\rangle}$

The point of tangency would be at $r\left(\frac{\pi}{4}\right) = \left(\sqrt{2}, \sqrt{2}, \frac{\pi}{4}\right)$

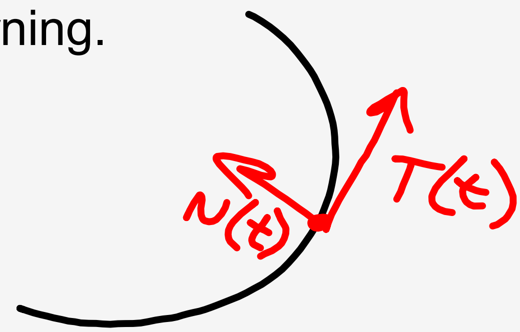
So the equations would be:

$$x(t) = \sqrt{2} - \sqrt{2}t, \quad y(t) = \sqrt{2} + \sqrt{2}t, \quad z(t) = \frac{\pi}{4} + t$$

If r(t) is a smooth curve, then the principal unit normal vector (pun v) is:

$$N(t) = \frac{T'(t)}{||T'(t)||}$$

A unit normal vector is simply any unit vector orthogonal to the unit tangent vector. However, the principal unit normal vector points in the direction that the curve is turning.



ex) Find the principal unit normal vector for

r(t) = < 2cost, 2sint, t >

$$T(t) = \frac{\langle -2\sin t, 2\cos t, 1\rangle}{\sqrt{5}} \quad \text{so} \quad \boxed{T'(t) = \frac{\langle -2\cos t, -2\sin t, 0\rangle}{\sqrt{5}}}$$

$$\cancel{N(t) = \frac{\frac{\langle -2\cos t, -2\sin t, 0\rangle}{\sqrt{5}}}{\frac{2}{\sqrt{5}}} = \frac{1}{2}\langle -2\cos t, -2\sin t, 0\rangle = \langle -\cos t, -\sin t, 0\rangle}$$

$$|T'(t)| = \frac{\sqrt{4\cos^2 t + 4\sin^2 t + 0}}{\sqrt{5}} = \frac{2}{\sqrt{5}}$$

$$N(t) = \frac{T'(t)}{|T'(t)|} = \frac{\frac{\langle -2\cos t, -2\sin t, 0\rangle}{\sqrt{5}}}{\frac{2}{\sqrt{5}}}$$

$$= \langle -\cos t, -\sin t, 0\rangle$$

Do you want your life to be easier?

If you do, remember that a unit normal vector is simply a vector that is orthogonal to the unit tangent vector.

For plane curves (curves in 2-space only), you can think of the unit normal vectors as simply being the vectors that are the 'opposite reciprocals' of the unit tangent vector. The principal unit normal vector is the one that points toward the concave side of the curve.

So if $T(t) = < t^2 + 6, 1/t >$, then $N(t) = < -1/t, t^2 + 6 >$ or $< 1/t, -(t^2 + 6) >$

For this problem, the correct $N(t)$ would be $< 1/t, -(t^2 + 6) >$

How did I know? I wasn't sure so I did T' in my head and realized that the 'y-component' was always negative. If you graph an original curve for this tangent vector, it works. :)

Find T(t) and N(t) if r(t) = < e^{4t}, e^{-4t}, 2t >

$$r'(t) = \langle 4e^{4t}, -4e^{-4t}, 2 \rangle$$

$$\|r'(t)\| = \sqrt{16e^{8t} + 16e^{-8t} + 4}$$

$$T(t) = \frac{\langle 4e^{4t}, -4e^{-4t}, 2 \rangle}{\sqrt{16e^{8t} + 16e^{-8t} + 4}} = \frac{\langle 2e^{4t}, -2e^{-4t}, 1 \rangle}{\sqrt{4e^{8t} + 4e^{-8t} + 1}}$$

$$\frac{d}{dt} \frac{2e^{4t}}{\sqrt{4e^{8t}+4e^{-8t}+1}} = \frac{8e^{4t}\sqrt{4e^{8t}+4e^{-8t}+1} - \cancel{2}e^{4t}\left(\cancel{\tfrac{1}{2}}\right)\left(4e^{8t}+4e^{-8t}+1\right)^{-\frac{1}{2}}\left(32e^{8t} - 32e^{-8t}\right)}{4e^{8t}+4e^{-8t}+1}$$

$$= \frac{8e^{4t}\left(4e^{8t}+4e^{-8t}+1\right) - e^{4t}\left(32e^{8t} - 32e^{-8t}\right)}{\left(4e^{8t}+4e^{-8t}+1\right)^{\frac{3}{2}}}$$

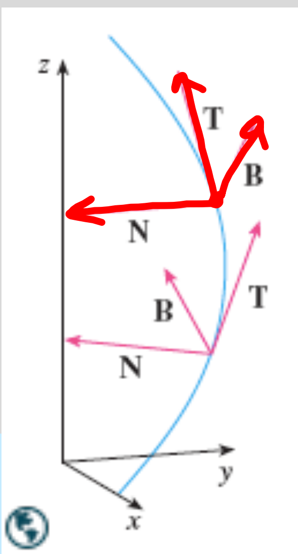
$$= \frac{\underline{32e^{12t}} + \underline{32e^{-4t}} + 8e^{4t} \underline{- 32e^{12t}} + 32e^{-4t}}{\left(4e^{8t}+4e^{-8t}+1\right)^{\frac{3}{2}}}$$

$$= \frac{64e^{-4t} + 8e^{4t}}{\left(4e^{8t}+4e^{-8t}+1\right)^{\frac{3}{2}}}$$

and so on

The 'Binormal Vector', called B(t), is a vector perpendicular to both T(t) and N(t).

So B(t) = T(t) x N(t) and all 3 of these vectors are orthogonal to each other.



The 3 of these vectors put together are called the TNB frame which moves along a curve with values of t. This gets used in differential geometry and has applications with motion in space.

A "Normal Plane" is a plane created by the vectors N(t) and B(t).

An "Osculating Plane" is a plane created by the vectors N(t) and T(t).

ex) Earlier we found T(t) and N(t) for r(t) = < 2cost, 2sint, t >. Use these to find the equations of the normal plane and osculating plane at the point P(0, 2, π/2).

$t = \frac{\pi}{2}$

Reminder: T(t) = (1/√5)<-2sint, 2cost, 1>

N(t) = <-cost, -sint, 0>

$T(\frac{\pi}{2}) = \langle -2, 0, 1 \rangle$

Since the equation of a plane requires a vector perpendicular to the plane, the normal plane equation would use T(t) as it's 'normal vector'. So at t = π/2, T(π/2) = (1/√5)<-2, 0, 1>. We can drop the (1/√5) for our plane equation so our equation would be:

$-2(x - 0) + 0(y - 2) + 1(z - \pi/2) = 0$

OR $-2x + z = \pi/2$

For the osculating plane, we would use B(t) for our normal vector.

B(π/2) = <-2, 0, 1> x <0, -1, 0> = <1, 0, 2>

So our equation would be:

$1(x - 0) + 0(y - 2) + 2(z - \pi/2) = 0$

OR $x + 2z = \pi$

What have we learned??

- Can I find a unit tangent vector, unit normal vector, and binormal vector on a space curve?
- Can I find the equation of a normal and osculating plane at a point on a space curve?

September 30, 2019