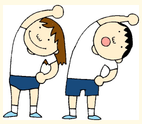
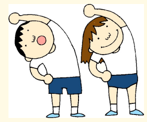


WARM UP!!



Find each of the following:

$\div$

$$1)\ \int \frac{x^3 - x + 3}{x^2 + x - 2}\,dx$$

partial fractions

$$1)\ \int \frac{x^3 - x + 3}{x^2 + x - 2}\,dx = \int \frac{x^3 - x + 3}{(x+2)(x-1)}\,dx = \int \left(x - 1 + \frac{2x+1}{(x+2)(x-1)} \right) dx$$

$$= \frac{1}{2}x^2 - x + \ln|x+2| + \ln|x-1| + C$$

$$2)\ \int \frac{\sqrt{1-x^2}}{x^4}\,dx = \int -\frac{\sin^2\theta}{\cos^4\theta}\,d\theta = \int -\sec^2\theta\tan^2\theta\,d\theta = -\frac{1}{3}\tan^3\theta + C = -\frac{(1-x^2)\sqrt{1-x^2}}{3x^3} + C$$

(trig substitution, letting $\sin\theta = \sqrt{1-x^2}$)

$$3)\ \int x \arcsin x\,dx = \frac{1}{2}x^2 \arcsin x - \int \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{1}{2}x^2 \arcsin x + \int \cos^2\theta\,d\theta$$

$$\frac{1}{2}x^2 \arcsin x + \int \frac{1}{2}(1 + \cos 2\theta)\,d\theta = \frac{1}{2}x^2 \arcsin x + \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{2}x^2 \arcsin x + \frac{1}{2}\arccos x - \frac{1}{2}x\sqrt{1-x^2} + C$$

trig sub

13.3a Arc Length and Curvature!

LEARNING TARGETS

At the end of this lesson, you will be able to:

- find the arc length of a space curve
- find the curvature of a curve at a point on the curve

Remember the formula for arc length of a curve defined parametrically?

$$s = \int_a^b speed\, dt = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

Relating this to our current content, we get:

$$s = \int_a^b ||r'(t)||dt = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}dt$$

Find the arc length of the curve given by r(t) = < 1, t^2, t^3> from t = 0 to t = 2

$$r'(t) = \langle 0, 2t, 3t^2 \rangle$$

$$s = \int_0^2 \sqrt{0 + 4t^2 + 9t^4}\,dt = \int_0^2 t\sqrt{4 + 9t^2}\,dt$$

let $u = 4 + 9t^2$ so $du = 18t\,dt$ $\quad \frac{1}{18}du = t\,dt$

$$s = \frac{1}{18}\int_4^{40} \sqrt{u}\,du = \frac{1}{18} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_4^{40} = \frac{1}{27}(80\sqrt{10} - 8)$$

Curvature - is the measure of how sharply a curve bends. It is the magnitude of the rate of change of the tangent vector with respect to the arc length (and is very similar in concept to the concavity of a plane curve).

$$K = \left\|\frac{dT}{ds}\right\| = \frac{\left\|\frac{dT}{dt}\right\|}{\left\|\frac{ds}{dt}\right\|} = \frac{\|T'(t)\|}{\|r'(t)\|} = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$

Find the curvature of $r(t) = < 2t, t^2, -1/3\ t^3 >$ at $\boxed{t = 1}$

$$r'(t) = \langle 2, 2t, -t^2 \rangle$$

$$r''(t) = \langle 0, 2, -2t \rangle$$

$$\rightarrow r'(1) = \langle 2, 2, -1 \rangle$$

$$\rightarrow r''(1) = \langle 0, 2, -2 \rangle$$

$$\begin{vmatrix} i & j & k \\ 2 & 2 & -1 \\ 0 & 2 & -2 \end{vmatrix} = \langle -2, 4, 4 \rangle$$

$$K = \frac{\sqrt{4+16+16}}{\left(\sqrt{4+4+1}\right)^3} = \frac{\sqrt{36}}{(\sqrt{9})^3} = \frac{6}{27} = \frac{2}{9}$$

r'

OR

$r'(t) =< 2, 2t, -t^2 >$ so $||r'(t)|| = \sqrt{4 + 4t^2 + t^4} = \sqrt{(t^2+2)^2} = t^2 + 2$

$r'(1) =< 2, 2, -1 >$ and $||r'(1)|| = \sqrt{9} = 3$

$$T(t) = \frac{< 2, 2t, -t^2 >}{t^2 + 2}$$

$$T'(t) = \frac{< 0, 2, -2t > (t^2+2) - < 2, 2t, -t^2 > (2t)}{(t^2+2)^2}$$

$$T'(1) = \frac{< 0, 2, -2 > (3) - < 2, 2, -1 > (2)}{9} = \frac{< -4, 2, -4 >}{9}$$

$$||T'(1)|| = \sqrt{\frac{16}{81} + \frac{4}{81} + \frac{16}{81}} = \frac{6}{9} = \frac{2}{3}$$

$$K = \frac{||T'(1)||}{||r'(1)||} = \frac{\left(\frac{2}{3}\right)}{3} = \frac{2}{9}$$

Find the curvature of r(t) = < 2t, t^2, -1/3 t^3 > at any t

$$r'(t)=\langle 2, 2t, -t^2\rangle$$

$$r''(t)=\langle 0, 2, -2t\rangle$$

$$r'\times r''=\begin{vmatrix} i & j & k \\ 2 & 2t & -t^2 \\ 0 & 2 & -2t \end{vmatrix}=\langle -2t^2, 4t, 4\rangle$$

$$K=\frac{\sqrt{4t^4+16t^2+16}}{\left(\sqrt{4+4t^2+t^4}\right)^3}=\frac{2\cancel{\sqrt{t^4+4t^2+4}}}{(t^4+4t^2+4)\cancel{\sqrt{t^4+4t^2+4}}}=\frac{2}{t^4+4t^2+4}$$

OR

$$r'(t)=<2,2t,-t^2> \quad so \ ||r'(t)||=\sqrt{4+4t^2+t^4}=\sqrt{(t^2+2)^2}=t^2+2$$

$$T(t)=\frac{<2,2t,-t^2>}{t^2+2}$$

$$T'(t)=\frac{<0,2,-2t>(t^2+2)-<2,2t,-t^2>(2t)}{(t^2+2)^2}=\frac{<-4t,-2t^2+4,-4t>}{(t^2+2)^2}$$

$$||T'(t)||=\frac{\sqrt{16t^2+4t^4-16t^2+16+16t^2}}{(t^2+2)^2}=\frac{\sqrt{4t^4+16t^2+16}}{(t^2+2)^2}=\frac{\sqrt{4(t^2+2)^2}}{(t^2+2)^2}$$

$$=\frac{2(t^2+2)}{(t^2+2)^2}=\frac{2}{t^2+2}$$

$$K=\frac{||T'(t)||}{||r'(t)||}=\frac{\left(\frac{2}{t^2+2}\right)}{t^2+2}=\frac{2}{(t^2+2)^2}$$

Curvature in rectangular coordinates

If C is the graph of a twice-differentiable function y = f(x), then the curvature K at the point (x, y) is given by:

$$K = \frac{|y''|}{[1 + (y')^2]^{\frac{3}{2}}}$$

The circle passing through (x, y) with radius $\boxed{r = 1/K}$ is called the "circle of curvature".

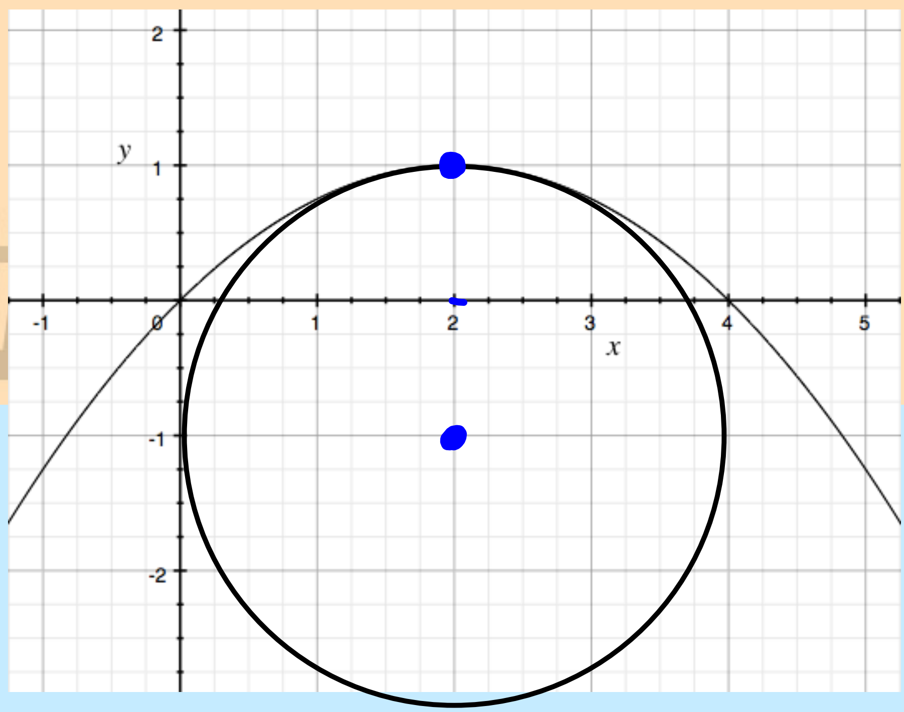
radius of curvature

Find the curvature of the parabola given by $y = x - (1/4)x^2$ at $x = 2$. Then find the radius and sketch the circle of curvature at (2, 1).

$y' = 1 - x/2, \quad y'' = -1/2$

$$K = \frac{1/2}{[1 + (1 - x/2)^2]^{3/2}} \qquad \text{so } K(2) = 1/2$$

The radius of the circle of curvature is 2.



What have we learned??

- Can I find the arc length of a space curve?
- Can I find the curvature of a curve at a point on the curve?