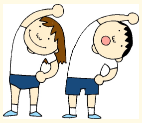
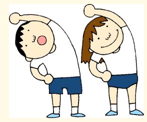


WARM UP!!

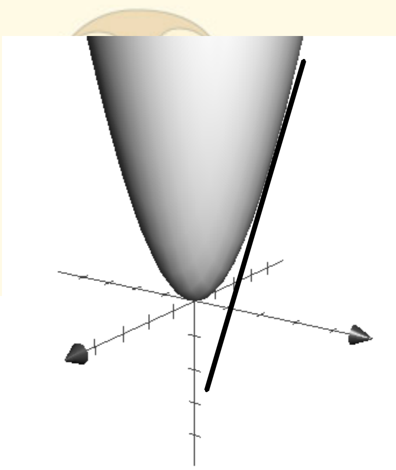


Imagine the surface $z = x^2 + y^2$

Find the equation of the plane tangent to the surface at the point (0, 3, 9).

Hint: If you can figure out the slope of the line tangent to the surface that passes through (0, 3, 9), then you can figure out a vector on the plane and use it to create an orthogonal vector. Feel free to use grapher to help get a visual.

Then find the distance from the origin to the plane.

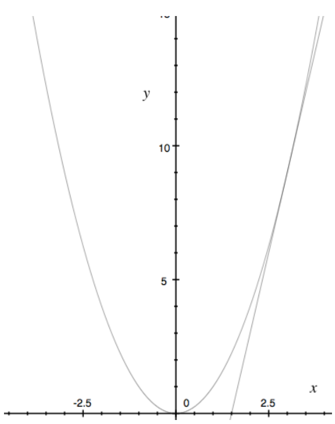


The graph looks like this. Since the plane passes through (0, 3, 9), if we let x = 0 we get $z = y^2$. So z' = 2y. At y = 3, the 'slope' would be 6. So a vector parallel to the plane would be <0, 1, 6> and a normal vector would be <0, -6, 1>.

The equation is:

$0(x - 0) - 6(y - 3) + 1(z - 9) = 0$

or $-6y + z = -9$ or $6y - z = 9$



For the distance, let v = <0, 3, 9>. The length of the projection of v onto <0, -6, 1> is $|(0 - 18 + 9)/\sqrt{37}| = 9/\sqrt{37}$

13.1 Vector-Valued Functions

ESSENTIAL LEARNING TARGETS
At the end of this lesson, you will be able to:

- analyze and sketch a space curve given by a vector-valued function
- extend the concepts of limits and continuity to vector-valued functions

A **vector-valued function** is a function where the output is a vector.

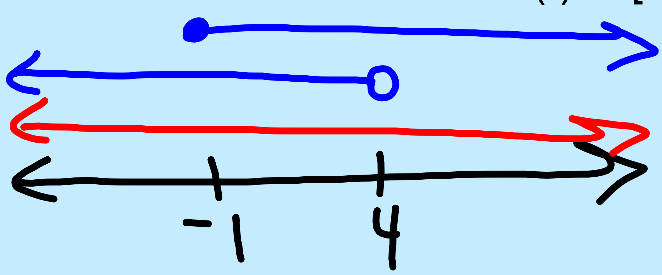
So r(t) = <f(t), g(t), h(t)> would be a vector-valued function where f(t), g(t) and h(t) would be the component functions.

The DOMAIN of a vector-valued function is the intersection of the domains of all of the component functions. (The set of all values of t for which every component function is defined.)

ex) State the domain of r(t) = <cost, ln(4 - t), $\sqrt{t + 1}$>

$4 - t > 0 \quad t < 4$

domain of cost is $(-\infty, \infty)$
domain of ln(4 - t) is $(-\infty, 4)$
domain of $\sqrt{t + 1}$ is $[-1, \infty)$
So the overall domain of r(t) is $[-1, 4)$



The graph of a vector-valued function r(t) = <f(t), g(t), h(t)> is simply the graph of the curve created by the parametric equations x = f(t), y = g(t) and z = h(t). The terminal point of each vector traces the curve in space. You do not need to be able to sketch the graph, just recognize or work with a given graph for a given function.

ex) Let's see what they look like.

- Open Grapher, and select 3D with a white background.
- In the lower right corner, click the "+" sign
- select the 2nd option down, "new equation from template">
- select "Cartesian surface"
- In the 'empty' matrix, enter each component for x, y, and z.
- You will not be using the u, only the t, so you might want to change t to go from -10 to 10 for each of these.

Sketch the graph of r(t) = <t, t + 2, 0>

Then sketch r(t) = <t, t + 2, t>

Then sketch r(t) = <sint, cost, 0>

Then sketch r(t) = <sint, cost, t>

To write a vector-valued function based on a function in 'rectangular' form, the process is exactly the same as writing parametric equations for the function.

ex) Write the plane curve $2x + 3y = 5$ as a vector-valued function.

The easiest way to write this is just to let $x = t$, then $y = (1/3)(5 - 2t)$.

So $r(t) = \langle t, (1/3)(5 - 2t) \rangle$

Limits of vector-valued functions are exactly the same as limits of regular functions, just find the limit of each component separately. Note, if the limit of one component DNE, then the entire limit DNE.

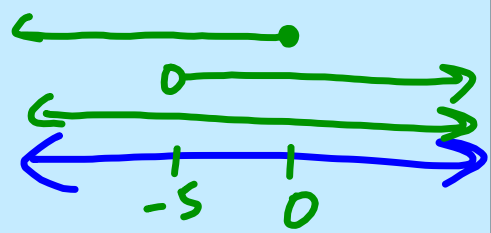
ex) Find $\lim_{t \to 0} \langle e^t, \frac{\sin t}{t}, e^{-t} \rangle$ = <1, 1, 1> ✓

$$\lim_{t \to 0} \frac{\sin t}{t} = \lim_{t \to 0} \frac{\cos t}{1} = 1$$

Continuity of vector-valued functions works the same as continuity for regular functions. The interval of continuity is the intersection of the interval of continuity for each component.

ex) Determine the interval on which the following vector-valued function is continuous:

$$r(t) = \langle e^{-t}, \ln(t+5), \sqrt{-t} \rangle$$



e^{-t} is continuous on $(-\infty, \infty)$

$\ln(t + 5)$ is continuous on $(-5, \infty)$

$\sqrt{(-t)}$ is continuous on $(-\infty, 0]$

So the interval of continuity is $(-5, 0]$

Find vector-valued functions forming the boundaries for the region sketched below. Assume each starts at t = 0. Give the interval of the parameter for each boundary.

look at bottom for figure

Note, there are many possible answers, but the requirement to start at t = 0 limits this a bit. You can't just make x = t unless the boundary starts at x = 0.

The easiest approach is to just find the equations of the lines in rectangular form, then let x = t. If you can choose any values for your t's, then just let t go from the lower x-value to the higher x-value.

The snag comes in when you have to deal with direction, or pre-assigned t-values. Here's how you deal with this:

If the t-values need to start at a specific number, you can 'shift' your t's by changing all t's to (t - #) or (# - t) where # is the # of spaces you need to shift. The (# - t) will also reverse your direction which can come in handy.

For this problem, the equations I get are:

$y = (3/2)x$, $y = 3$, and $y = (-3/2)x$

So my initial vector-valued functions would be:

$r_1(t) = < t, (3/2)t >$ for $0 \le t \le 2$

$r_2(t) = < t, 3 >$ for $-2 \le t \le 2$

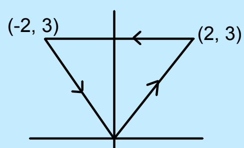
$r_3(t) = < t, (-3/2)t >$ for $-2 \le t \le 0$

These all generate the correct line segments, but they don't all meet the criteria of t starting at 0 and they aren't all in the correct direction. r_1 is fine. r_2 needs to be shifted and have the direction reversed. r_3 just needs to be shifted. For r_2, change the t to (2 - t) and have the ts go from 0 to 4 (a shift of 2 to move the interval up to start at 0, and a reversal to change the direction). This way when t = 0, 2 - t = 2 and when t = 4, 2 - t = -2. For r_3, we don't want to reverse the direction, just shift, so change t to (t - 2) and have the ts go from 0 to 2. This way when t = 0, we are at the point (-2, (-3/2)(-2)) = (-2, 3), and when t = 2 we are at the point (0, 0). So our final equations would be:

$r_1(t) = < t, (3/2)t >$ for $0 \le t \le 2$

$r_2(t) = < 2 - t, 3 >$ for $0 \le t \le 4$

$r_3(t) = < t - 2, (-3/2)(t - 2) >$ for $0 \le t \le 2$



$r_1 = \langle t, \frac{3}{2}t \rangle$, t in [0,2]

$r_2 = \langle t, 3 \rangle$, t in [-2,2]

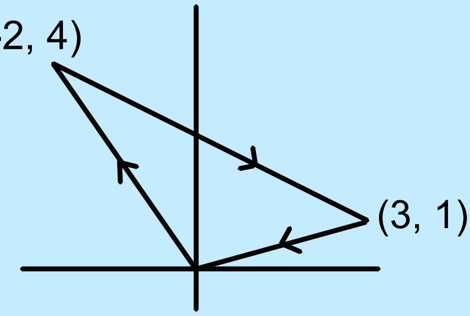
$r_3 = \langle t, -\frac{3}{2}t \rangle$, t in [-2,0]

$r_2 = \langle 2-t, 3 \rangle$, t in [0,4]

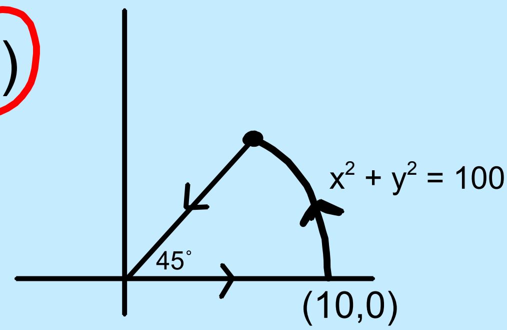
$r_3 = \langle t-2, -\frac{3}{2}(t-2) \rangle$, t in [0,2]

Find vector-valued functions for the boundaries of each region below. Assume each starts at t = 0. Give the interval of the parameter for each boundary.

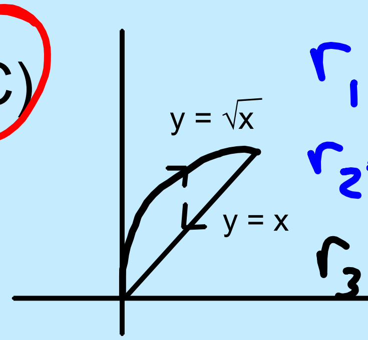
A)



B)



C)



$r_1 = \langle t, 0\rangle$, t in $[0, 10]$

$r_2 = \langle t, \sqrt{100-t^2}\rangle$, t in $[\frac{10}{\sqrt{2}}, 10]$

$r_3 = \langle t, t\rangle, [0, \frac{10}{\sqrt{2}}]$

$= \langle -t, -t\rangle, [0, \frac{10}{\sqrt{2}}]$

A) $r_1(t) = \langle -t, 2t\rangle, 0 \le t \le 2$

$r_2(t) = \langle t - 2, (-3/5)(t - 5) + 1\rangle\ 0 \le t \le 5$

$r_3(t) = \langle 3(1 - t), 1 - t\rangle, 0 \le t \le 1$

B) $r_1(t) = \langle t, 0\rangle, 0 \le t \le 10$ ✓

$r_2(t) = \langle 10 - t, \sqrt{(100 - (10 - t)^2)}\rangle, 0 \le t \le 10 - 5\sqrt{2}$

OR an easier way is:

$r_2(t) = \langle 10\cos t, 10\sin t\rangle, 0 \le t \le 45^\circ$ ✓

$r_3(t) = \langle 5\sqrt{2} - t, 5\sqrt{2} - t\rangle, 0 \le t \le 5\sqrt{2}$

C) $r_1(t) = \langle t, \sqrt{t}\rangle, 0 \le t \le 1$

$r_2(t) = \langle 1 - t, 1 - t\rangle, 0 \le t \le 1$

What have we learned??

- Can I analyze a space curve given by a vector-valued function?
- Can I extend the concepts of limits and continuity to vector-valued functions?

September 24, 2019