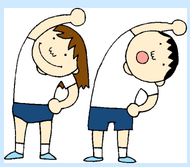
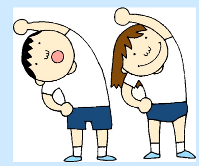


WARM UP!!



1) Suppose u = <-2, 1, 4>, v = <3, 2, -1> and w = <1, -1/2, -2>
   a) Find the length of v
   b) Find the unit vector in the direction of v
   c) Are any of the vectors parallel?

2) Find three other points that are collinear with (1, 0, -2) and (2, -2, 1)

u = <1, -2, 3> direction vector

3) A diameter of a sphere has endpoints (1, 3, -2) and (5, -1, 2).
Find the equation of the sphere.

1) a) $\|v\| = \sqrt{9 + 4 + 1} = \sqrt{14}$ b) unit vector = $\langle 3, 2, -1 \rangle / \sqrt{14}$
   c) u = -2w so u and w are parallel

2) direction vector = <1, -2, 3> so adding this to the second point we get (3, -4, 4), (4, -6, 7) and (5, -8, 10)

3) center: (3, 1, 0) diameter: $\sqrt{16 + 16 + 16} = 4\sqrt{3}$ so radius is $2\sqrt{3}$

equation: $(x - 3)^2 + (y - 1)^2 + z^2 = 12$

12.3 Dot Products!!

ESSENTIAL LEARNING TARGETS
At the end of this lesson, you will be able to:

- calculate the dot product of 2 vectors
- use dot products to find the angle between 2 vectors
- find the direction cosines of a vector in space

Dot Product

If **u** = $\langle u_1, u_2 \rangle$ and **v** = $\langle v_1, v_2 \rangle$, then

$u \cdot v = u_1v_1 + u_2v_2$

If **u** = $\langle u_1, u_2, u_3 \rangle$ and **v** = $\langle v_1, v_2, v_3 \rangle$, then

$u \cdot v = u_1v_1 + u_2v_2 + u_3v_3$

Properties of dot products

1) $u \cdot v = v \cdot u$
2) $u \cdot (v + w) = u \cdot v + u \cdot w$
3) $c(u \cdot v) = cu \cdot v = u \cdot cv$
4) $0 \cdot v = 0$
5) $v \cdot v = |v|^2$

$3(4 \cdot 5)$

Suppose u = <2, -2>, v = <5, 8> and w = <-4, 3>

Find:

a) $u \cdot v$ = 2(5) + -2(8) = -6 ✓

b) $(u \cdot v)w$ = -6<-4, 3> = <24, -18>

c) $u \cdot (2v)$ = 2(10) + -2(16) = -12

d) $||w||^2$ = $w \cdot w$ = -4(-4) + 3(3) = 25

Angle Between Vectors

If θ is the angle between 2 nonzero vectors u and v, then

$$\cos\theta = \frac{u \cdot v}{\|u\| \, \|v\|}$$

This can be rewritten to get $u \cdot v = \|u\| \, \|v\| \cos\theta$

Based on this, if u • v = 0, this means that cosθ = 0, which means the vectors must be: orthogonal (perpendicular)

ex) Find the angle between u = <3, -1, 2> and v = <-4, 0, 2>

✓ $$\cos\theta = \frac{3(-4) + -1(0) + 2(2)}{\sqrt{9+1+4}\sqrt{16+0+4}} = \frac{-8}{\sqrt{14}\sqrt{20}} = \frac{-8}{\sqrt{280}} = \frac{-4}{\sqrt{70}}$$

$$\theta = \arccos\left(\frac{-4}{\sqrt{70}}\right) \approx 118.561°$$

Orthogonal, Parallel or Neither?

ex) u = -2i + 3j - k and v = 2i + j - k

✓ u • v = -2(2) + 3(1) - 1(-1) = 0 -> orthogonal

ex) u = -1/3 (i - 2j) and v = 2i - 4j

✓ Since v = -6u, the vectors are parallel

Direction Cosines

In a plane, we measure the direction of the vector by finding the angle between the positive x-axis and the vector.

In 3-dimensional space it works better to measure the angle between the vector and each of the axes. We do this by actually finding the angles between the vector and the 3 unit vectors i, j, and k. These are called the **direction angles**.

If α is the angle between vector, v, and unit vector, i, then

$$\cos\alpha = \frac{v \bullet i}{||v||\,||i||} = \frac{\langle v_1, v_2, v_3\rangle \bullet \langle 1, 0, 0\rangle}{||v||(1)} = \frac{v_1}{||v||}$$

Similary, if β is the angle between v and j, and γ is the angle between v and k, then

$$\cos\beta = \frac{v_2}{||v||} \quad \text{and} \quad \cos\gamma = \frac{v_3}{||v||}$$

One major property of direction cosines:

Since $\frac{v}{||v||}$ is the unit vector in the direction of v,

and since $\frac{v}{||v||} = \frac{v_1}{||v||} i + \frac{v_2}{||v||} j + \frac{v_3}{||v||} k = \cos\alpha\, i + \cos\beta\, j + \cos\gamma\, k$

this means that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

ex) Find the direction cosines and angles for v = 2i + 3j + 4k, and show that the property above is true.

$||v|| = \sqrt{4+9+16} = \sqrt{29}$

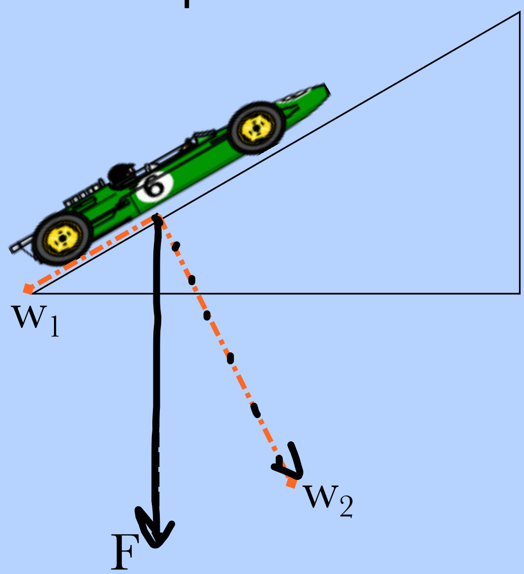
$\cos\alpha = \frac{2}{\sqrt{29}}$ so $\alpha \approx 68.199°$

✓ $\cos\beta = \frac{3}{\sqrt{29}}$ so $\beta \approx 56.145°$

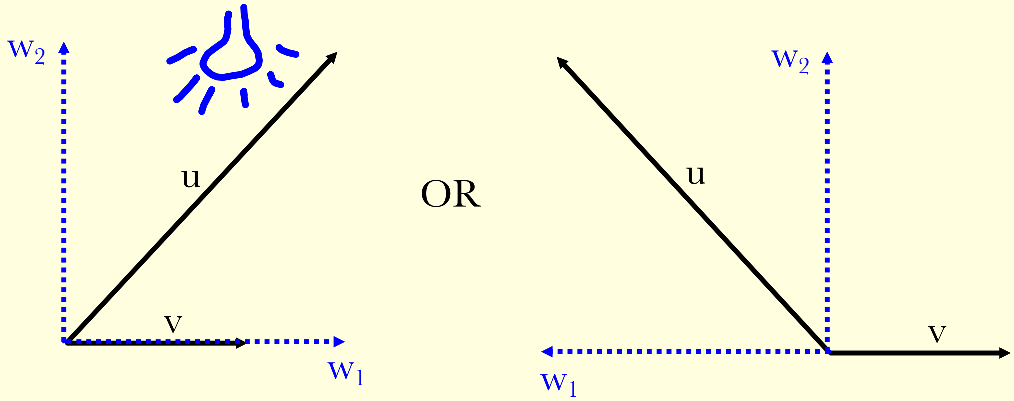
$\cos\gamma = \frac{4}{\sqrt{29}}$ so $\gamma \approx 42.031°$

$\frac{4}{29} + \frac{9}{29} + \frac{16}{29} = \frac{29}{29} = 1$

So far we have learned how to take several vectors and combine them to find a resultant vector. Now we are going to learn how to break a vector apart into orthogonal components.



The forces w_1 and w_2 help you analyze the effect of gravity on the car. w_1 indicates the force pulling the car backwards and w_2 indicates the force that the tires must withstand.



Let u = $w_1 + w_2$, where w_1 is parallel to v and w_2 is orthogonal to v.

$w_1 = \text{proj}_v u$, is called the **projection of u onto v**

w_1 is also called the **vector component of u along v**

$w_2 = u - w_1$ and w_2 is called the **vector component of u orthogonal to v**

$$\text{proj}_v u = \left(\frac{u \cdot v}{||v||^2}\right) v = \frac{u \cdot v}{||v||} \frac{v}{||v||}$$

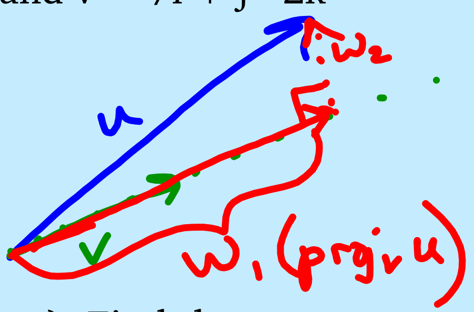
Great way to think of it. Really, w_1 and w_2 are just the horizontal and vertical components of the vector. Normally, when we write a vector in trig form, we know the angle of the vector. Projections allow us to find these components without trig. Remember that the length of a horizontal component is rcosθ, and

$\cos\theta = \dfrac{u \cdot v}{||u||\ ||v||}$, and ||u|| is the same thing as r, so the magnitude of the horizontal component is: $||u|| \dfrac{u \cdot v}{||u||\ ||v||} = \dfrac{u \cdot v}{||v||}$

Note: your book calls the magnitude of the projection the "scalar projection"

Multiplying this by v / ||v|| (the unit vector in the direction of v) gives us a vector in the direction of v whose length is the horizontal component of u.

ex) Find the projection of u onto v and the vector component of u orthogonal to v for the vectors: u = 3i - 5j + 2k and v = 7i + j - 2k



$$\text{proj}_v u = \left(\frac{u \cdot v}{||v||^2}\right) v$$

$$\text{proj}_v u = \left(\frac{3(7) - 5(1) + 2(-2)}{49 + 1 + 4}\right)\langle 7, 1, -2\rangle$$

$$= \left(\frac{12}{54}\right)\langle 7, 1, -2\rangle = \frac{2}{9}\langle 7, 1, -2\rangle = \frac{14}{9}i + \frac{2}{9}j - \frac{4}{9}k$$

So if $w_1 = \frac{14}{9}i + \frac{2}{9}j - \frac{4}{9}k$ and $w_2 = u - w_1$

$$w_2 = \frac{13}{9}i - \frac{47}{9}j + \frac{22}{9}k$$

ex) Find the vector component of u = <7, 4> that is orthogonal to v = <2, 3>, given that $w_1 = \text{proj}_v u$ = <4, 6> and $u = w_1 + w_2$

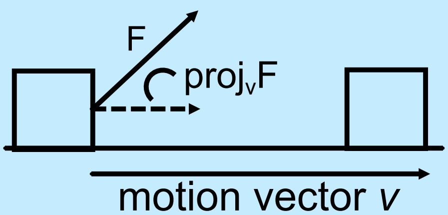
✔ $w_2 = u - w_1$ = <7, 4> - <4, 6> = <3, -2>

Check: <2, 3> • <3, -2> = 0

WORK!!

The work done by a constant force along a line (vector) of motion of an object = the magnitude of the force times the distance the object moved, so the work equals the magnitude of the force times the magnitude of the motion vector

But what if the force is not directed along the line of motion? Then the work done is the projection of the force onto the vector times the magnitude of the motion vector



$$Work = ||proj_v F||\,||v|| = \left\|\left(\frac{F \cdot v}{||v||^2}\right) v\right\| (||v||) = \frac{F \cdot v}{||v||^2}||v||\,||v|| = F \cdot v$$

So $W = ||proj_v F||\,||v||$

or $W = F \cdot v$

force vector (direction)

vector of motion

ex) To close a sliding barn door, a farmer pulls on a rope with a constant force of 50 pounds at a constant angle of 60˚. Find the work done in moving the door 12 feet.

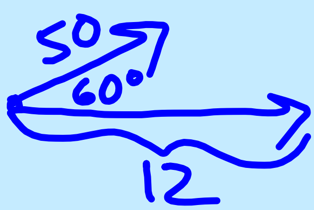
Remember that $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$

work = $\mathbf{F} \cdot \mathbf{v} = \|\mathbf{F}\|\|\mathbf{v}\|\cos\theta = (50)(12)\cos 60^\circ$

$600(1/2) = 300$ foot-pounds

$\langle 50\cos 60, 50\sin 60 \rangle$

$\langle 12, 0 \rangle$



September 9, 2019

Review!

1) An airplane is heading N35˚W with an airspeed of 376 mph. It encounters a wind of 41mph in the direction of N21˚E. What are the resultant speed (with respect to the ground) and direction of the plane?

2) Forces with magnitudes of 500 pounds and 200 pounds act on a machine part at angles of 30˚ and -45˚ respectively with the x-axis. Find the direction and magnitude of the resultant force.

1) Angle of plane = 125˚, so p = $\langle 376\cos 125^\circ, 376\sin 125^\circ \rangle$

angle of wind = 69˚, so w = $\langle 41\cos 69^\circ, 41\sin 69^\circ \rangle$

resultant = $\langle 376\cos 125^\circ + 41\cos 69^\circ, 376\sin 125^\circ + 41\sin 69^\circ \rangle$

$\approx \langle -200.971, 346.278 \rangle$

$||\text{resultant}|| \approx 400.372$ mph

$\theta \approx \tan^{-1}(346.278/-200.971) \approx -59.870^\circ$ or <u>120.130°</u>

2) $F_1 = 500(\cos 30^\circ, \sin 30^\circ)$

$F_2 = 200(\cos(-45^\circ), \sin(-45^\circ))$

$R = \langle 500\cos 30^\circ + 200\cos(-45^\circ), 500\sin 30^\circ + 200\sin(-45^\circ) \rangle$

$\approx \langle 574.434, 108.579 \rangle$

$||R|| \approx 584.605$ pounds

$\theta \approx \tan^{-1}(108/579/574.434) \approx 10.704^\circ$

What have we learned??

- Can I calculate the dot product of 2 vectors?
- Can I use dot products to find the angle between 2 vectors?
- Can I find the direction cosines of a vector in space?