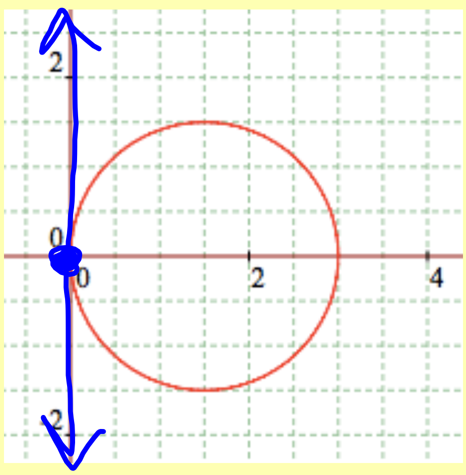
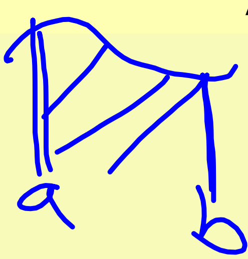
WARMUP!!

ex) Graph $r = 3\cos\theta$. Then find the area bounded by the graph using any method.



Area = $\pi(3/2)^2 = 9/4\ \pi$



10.5a - Area of Polar Graphs!

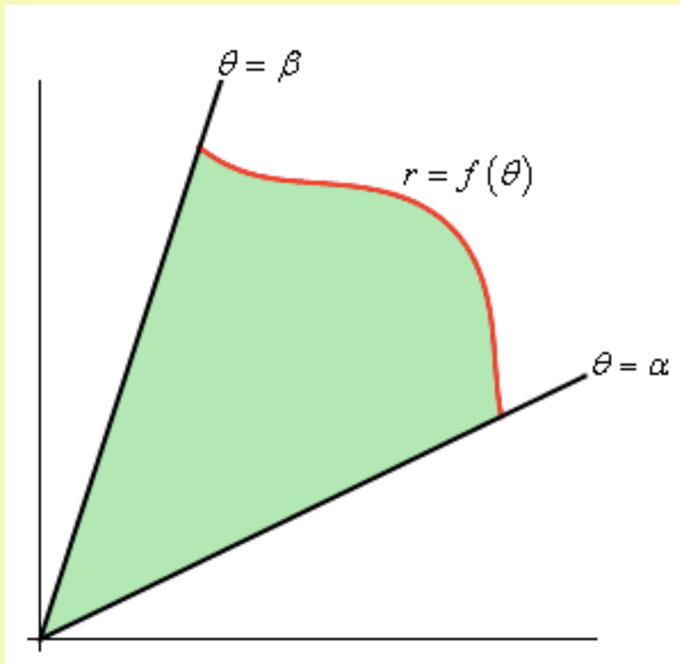
At the end of this lesson the students will be able to:

- find the area of a graph bounded by a polar equation

Of course we know that the area of a sector of a circle is given by: $A = (1/2)\theta r^2$

The area of a polar region is simply the sum of infinitely many little sectors added together.

α and β are the radial lines that bound the region. To find these, you will need to determine the angles that begin and end the portion of the graph you are interested in.



Area of a Polar Region: $$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$

ex) Find the area bounded by the graph of r = 3cosθ.

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

Because the graph begins and ends with a radius of 0, you can find the angles by setting r = 0 and solving. Note that you want a continuous interval where the graph begins and ends with 0 without doubling over itself, so be careful when choosing which angles to use.

For our problem:

$$3\cos\theta = 0$$

$$\cos\theta = \cdots, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

If we sketch the graph (which we did in the warmup), we will find that we can use any of these intervals as long we pick consecutive values.

$$A = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (3\cos\theta)^2 d\theta$$

$$= \frac{9}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta \, d\theta$$

$$= \frac{9}{2}\cdot\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos 2\theta) d\theta$$

$$= \frac{9}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{9}{4}\left(\frac{\pi}{2} + 0 - \left(-\frac{\pi}{2} + 0\right)\right) = \frac{9}{4}\pi$$

You try) Find the area of one petal of r = 6sin2θ

After sketching, I chose the interval [0, π/2].

$$A = \frac{1}{2}\int_{0}^{\frac{\pi}{2}} (6\sin 2\theta)^2\, d\theta = 18\int_{0}^{\frac{\pi}{2}} \sin^2 2\theta\, d\theta$$

$$= 9\int_{0}^{\frac{\pi}{2}} (1 - \cos 4\theta) d\theta = 9\left(\theta - \frac{1}{4}\sin 4\theta\right)\Bigg|_{0}^{\frac{\pi}{2}}$$

$$= \frac{9\pi}{2} - 0 - (0 - 0) = \frac{9\pi}{2}$$

ex) Find the points of intersection of the graphs of the equations r = 1 + cosθ and r = 3cosθ

1) Find the points of intersection by equating the radii.

$$1 + \cos\theta = 3\cos\theta$$

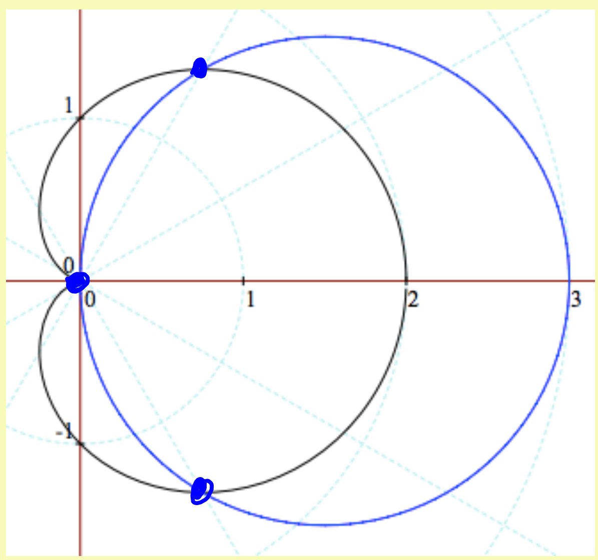
$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\left(\frac{3}{2}, \frac{\pi}{3}\right), \left(\frac{3}{2}, \frac{5\pi}{3}\right)$$

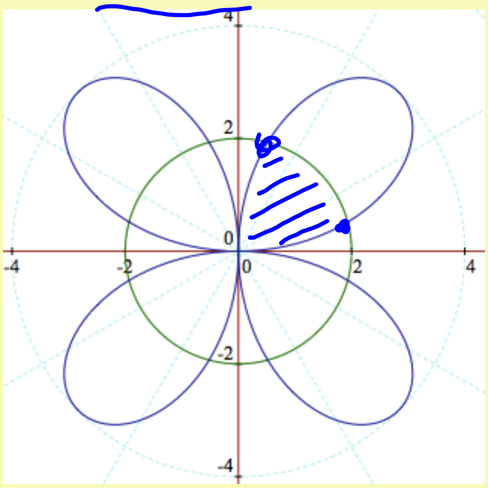
2) Graph the equations. Did we find all of the points? How can this be?

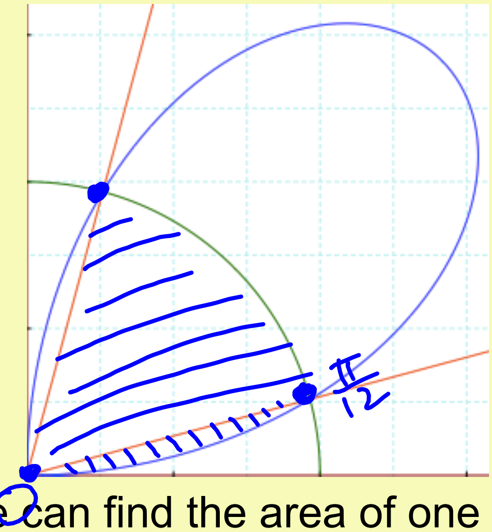
The problem with polar graphs is that the same point on a graph can be represented by different coordinates. (0, π) is the same as (0, π/2). The only way to truly get all points is to rewrite the polar equations in rectangular form, solve the system, and then convert the rectangular coordinates back to polar.



You will not be required to do this. All points of intersection will either be given, be found analytically with the polar equations, or be clearly evident based on a simple graph.

ex) Find the overlapping area of r = 2 and r = 4sin2θ.





Because of symmetry, we can find the area of one section and multiply it by 4. To find the points of intersection, equate the r's.

When you solve the equation 4sin2θ=2 you get θ = π/12, 5π/12, ...

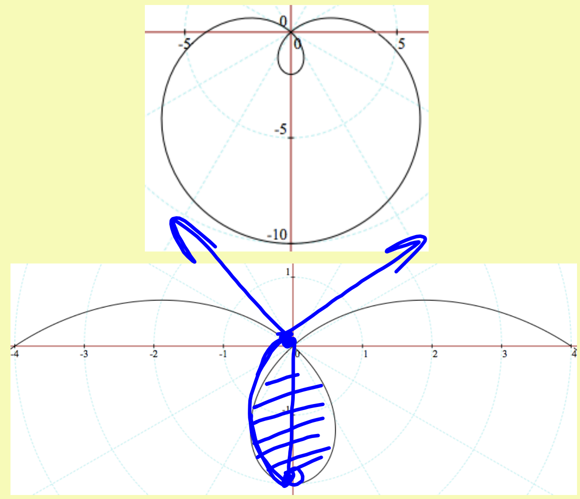
If you find the area of the sector of the circle from π/12 to 5π/12, you will miss small portions of the petals on either side. So we will need to find the area of these slivers by finding the area of the petal from 0 to π/12 (which you can double to get both slivers).

$$A = 4\left[\frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} (2)^2 d\theta + 2\left(\frac{1}{2}\right)\int_{0}^{\frac{\pi}{12}} (4\sin(2\theta))^2 d\theta\right]$$

$$= 4\left(\frac{4\pi}{3} - \sqrt{3}\right) = \frac{16\pi}{3} - 4\sqrt{3}$$

You try) Find the area of the inner loop of $r = 4 - 6\sin\theta$. (no calculator) 😈

θ	R = 4 – 6sinθ
0	4
π/6	1
π/4	$4 - 3\sqrt{2} \approx -1/4$
π/3	$4 - 3\sqrt{3} \approx -1$
π/2	-2
2π/3	$4 - 3\sqrt{3} \approx -1$
3π/4	$4 - 3\sqrt{2} \approx -1/4$
5π/6	1
π	4



To find the limits (angles), we need to figure out where the radius is 0. Setting r = 0 gives us θ = arcsin(2/3). Rather than trying to find the whole area, we can just integrate from arcsin(2/3) to π/2 and double it.

$$A = 2 \cdot \frac{1}{2} \int_{\arcsin\left(\frac{2}{3}\right)}^{\frac{\pi}{2}} (4 - 6\sin\theta)^2 d\theta$$

$$= \int_{\arcsin\left(\frac{2}{3}\right)}^{\frac{\pi}{2}} (16 - 48\sin\theta + 36\sin^2\theta)\, d\theta$$

$$= \int_{\arcsin\left(\frac{2}{3}\right)}^{\frac{\pi}{2}} (16 - 48\sin\theta + 18(1 - \cos(2\theta))\, d\theta$$

$$= 16\theta + 48\cos\theta + 18\theta - 9\sin(2\theta)\Big|_{\arcsin\left(\frac{2}{3}\right)}^{\frac{\pi}{2}}$$

$$= 8\pi + 0 + 9\pi - 0 - \left(16\arcsin\left(\frac{2}{3}\right) + 16\sqrt{5} + 18\arcsin\left(\frac{2}{3}\right) - 4\sqrt{5}\right)$$

$$= 17\pi - 12\sqrt{5} - 34\arcsin\left(\frac{2}{3}\right) \approx 1.7635$$

What have we learned?

- Can I find the area of a region on a polar graph?