WARMUP!!

2011 #1

Solution

10.3b - Concavity and Arc Length with Parametric Equations!

Essential Learning Target:

- I can use definite integrals to compute the length of a planar curve defined by a function or by a parametrically defined curve.

Higher order derivatives of parametrics.

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dt}\left[\frac{dy}{dx}\right] \cdot \frac{dt}{dx} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$$

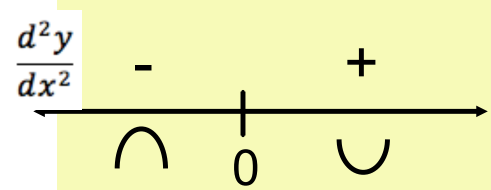
$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{d^2y}{dx^2}\right] = \frac{d}{dt}\left[\frac{d^2y}{dx^2}\right] \cdot \frac{dt}{dx} = \frac{\frac{d}{dt}\left[\frac{d^2y}{dx^2}\right]}{\frac{dx}{dt}}$$

Determine the intervals of t on which the curve is concave up or down if $x = 2 + t^2$ and $y = t^2 + t^3$

$$\frac{dx}{dt} = 2t \quad and \quad \frac{dy}{dt} = 2t + 3t^2$$

$$\frac{dy}{dx} = \frac{2t + 3t^2}{2t} = \frac{2 + 3t}{2} = 1 + \frac{3}{2}t$$

$$\frac{d^2y}{dx^2} = \frac{\frac{3}{2}}{2t} = \frac{3}{4t}$$



So the curve is concave down for t in $(-\infty, 0)$ because $d^2y/dx^2 < 0$ and concave up for t in $(0, \infty)$ because $d^2y/dx^2 > 0$.

You try! Ditto for $x = t^2$ and $y = \ln t$

$$\frac{dx}{dt} = 2t \quad \text{and} \quad \frac{dy}{dt} = \frac{1}{t}$$

$$\frac{dy}{dx} = \frac{\frac{1}{t}}{2t} = \frac{1}{2t^2}$$

$$\frac{d^2y}{dx^2} = \frac{-\frac{1}{t^3}}{2t} = -\frac{1}{2t^4}$$

This is always negative so the curve is concave down for $(0, \infty)$.

Whoa! Why not include all of the negative values of t?

Brain Break! It's Mathematician Monday!!

Maryam Mirzakhani was born May 3, 1977 in Tehran, Iran. In 1994 she was the first female Iranian student to take part in the International Mathematical Olympiads. In 1995 she competed again, becoming the first Iranian student to achieve a perfect score and win two gold medals. Maryan earned her bachelor's from Sharif University in Tehran in 1999 and her PhD from Harvard in 2004.

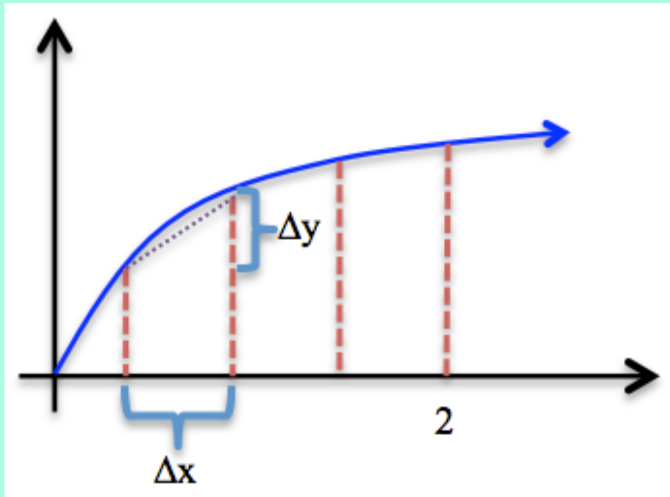
Maryan created a new proof for the formula discovered by Edward Witten and Maxim Kontsevich on the intersection numbers of tautological classes on moduli space, as well as an asymptotic formula for the growth of the number of simple closed geodesics on a compact hyperbolic surface, generalizing the theorem of the three geodesics for spherical surfaces. Her subsequent work has focused on Teichmüller dynamics of moduli space. In particular, she was able to prove the long-standing conjecture that William Thurston's earthquake flow on Teichmüller space is ergodic.

In 2014 Maryam Mirzakhani was the first ever female mathematician to receive the Fields Medal for her outstanding contributions to the dynamics of geometry of Riemann surfaces and their moduli spaces. At the ceremony, the presenter said the following:

"*Her work expertly blends dynamics with geometry. Among other things, she studies billiards. But now, in a move very characteristic of modern mathematics, it gets kind of meta: She considers not just one billiard table, but the universe of all possible billiard tables. And the kind of dynamics she studies doesn't directly concern the motion of the billiards on the table, but instead a transformation of the billiard table itself, which is changing its shape in a rule-governed way; if you like, the table itself moves like a strange planet around the universe of all possible tables ... This isn't the kind of thing you do to win at pool, but it's the kind of thing you do to win a Fields Medal. And it's what you need to do in order to expose the dynamics at the heart of geometry; for there's no question that they're there.*"

Maryam is now a mathematics Professor at Stanford University and is considered one of the greatest living female mathematicians. In a recent newspaper article Maryam described how her love of mathematics grew from her teenage years onwards: *'The more time I spent on maths, the more excited I got.'*

Do you remember when we derived the formula for arc length?

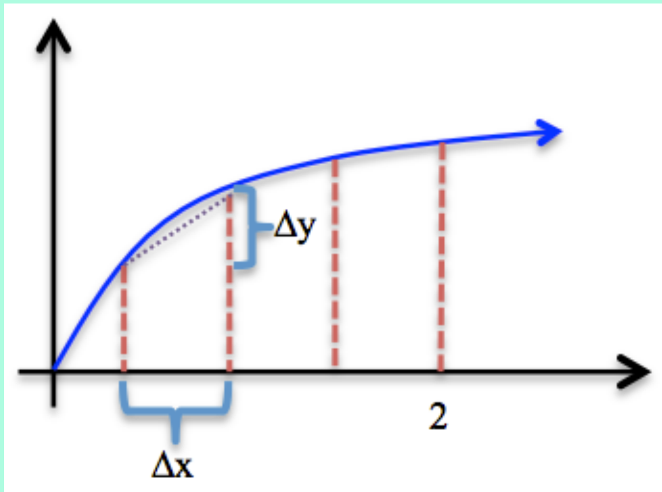


- Approximate length of single arc: $\sqrt{(\Delta x)^2 + (\Delta y)^2}$
- Simplify to: $\Delta x\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$
- The limit as each of these intervals becomes super small would give us: $dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$
- So to get the entire arc, take the sum of each length to get...

ARC LENGTH:

$$S = \int_a^b \sqrt{1 + [f'(x)]^2}dx$$

Can we derive an updated formula for arc length using parametrics?



- Approximate length of single arc: $\sqrt{(\Delta x)^2 + (\Delta y)^2}$

- The limit as each of this interval becomes extremely small gives: $\sqrt{(dx)^2 + (dy)^2}$

- Let's introduce some dts! $dt\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$\frac{dt}{dt}$

- Adding them all together gives us:

Arc Length: $$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Hmmm, what does this remind you of?

So the formula for arc length with parametric equations is:

$$S=\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

ex) Find the arc length of the curve with the equations $x = t^2 + 1$ and $y = 4t^3 + 3$ on the interval $-1 \le t \le 0$.

Side note: $\sqrt{t^2}=t$ if $t > 0$

$\sqrt{t^2}=-t$ if $t < 0$

$$\frac{dx}{dt}=2t \text{ and } \frac{dy}{dt}=12t^2$$

$u = 1+36t^2$

$$S=\int_{-1}^{0}\sqrt{(2t)^2+(12t^2)^2}\,dt=\int_{-1}^{0}\sqrt{4t^2+144t^4}\,dt$$

$$=\int_{-1}^{0}-2t\sqrt{1+36t^2}\,dt=-\frac{2}{72}\int_{37}^{1}u^{\frac{1}{2}}du=\frac{1}{36}\int_{1}^{37}u^{\frac{1}{2}}du$$

$$=\frac{1}{36}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_{1}^{37}=\frac{1}{54}\sqrt{37^3}-\frac{1}{54}$$

What have we learned?

- Can I find a higher order derivative for a set of parametric equations and use this to determine intervals of concavity?
- Can I calculate the arc length of a curve defined parametrically?
- Can I calculate the area of a surface formed by revolving a set of parametric equations about the x- or y-axis?

Let's finish up those quizzes!