WARMUP!!

Use any method to find dy/dx if:

$$x = \sqrt[3]{t} \quad \text{and} \quad y = 4 - t$$

Answer must be written in terms of t.

$$t = x^3, so\ y = 4 - x^3$$

$$\frac{dy}{dx} = -3x^2 = -3\left(\sqrt[3]{t}\right)^2 = \boxed{-3t^{\frac{2}{3}}}$$

OR

$$\frac{dx}{dt} = \frac{1}{3}t^{-\frac{2}{3}}$$

$$\frac{dy}{dt} = -1$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-1}{\frac{1}{3}t^{-\frac{2}{3}}} = \boxed{-3t^{\frac{2}{3}}}$$

10.3a - Derivatives of Parametric Equations!

Essential Learning Targets:

- I can extend methods of calculating derivatives of real-valued functions to vector-valued functions, parametric functions, and functions in polar coordinates.
- I can use derivatives to determine velocity, speed and acceleration for a particle moving along curves given by parametric or vector-valued functions.
- I can use definite integrals to determine displacement, distance and position of a particle moving along a curve given by parametric or vector-valued functions.

Try this!

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t for the parametric equations below. Then find the slope and concavity at t = 0.

$$x = t^2 + 3t + 2 \quad \text{and} \quad y = 2t$$

$$\frac{dy}{dt} = 2 \quad \text{and} \quad \frac{dx}{dt} = 2t + 3$$

$$\frac{dy}{dx} = \frac{2}{2t+3}$$

$$\left.\frac{dy}{dx}\right|_{t=0} = \frac{2}{3}$$

slope

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\text{the second derivative with respect to t}}{\frac{dx}{dt}}$$

$$\frac{d^2y}{dx^2} = \frac{-2(2t+3)^{-2}(2)}{2t+3} = \frac{-4}{(2t+3)^3}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=0} = -\frac{4}{27}$$

Let's try one with trig!

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of θ for the parametric equations below. Then find the slope and concavity at θ = 0.

x = cosθ and y = 3sinθ

$$\frac{dy}{d\theta} = 3\cos\theta \quad \text{and} \quad \frac{dx}{d\theta} = -\sin\theta$$

$$\frac{dy}{dx} = \frac{3\cos\theta}{-\sin\theta} = -3\cot\theta$$

$$\left.\frac{dy}{dx}\right|_{\theta=0} \text{ is undefined, so the slope is undefined}$$

$$\frac{d^2y}{dx^2} = \frac{3\csc^2\theta}{-\sin\theta} = -3\csc^3\theta$$

$$\left.\frac{d^2y}{dx^2}\right|_{\theta=0} \text{ is undefined, so the concavity is also undefined}$$

Try this one!

Find all points (if any) of horizontal and vertical tangency to the curve with the equations:

$x = t^2 - t$ and $y = t^3 - 3t$

$$\frac{dy}{dt} = 3t^2 - 3 \quad and \quad \frac{dx}{dt} = 2t - 1$$

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t - 1}$$

$$\frac{dy}{dx} = 0 \text{ at } t = \pm 1$$

$$\frac{dy}{dx} \text{ is undefined at } t = \frac{1}{2}$$

Note: Without finding dy/dx, the tangent line is horizontal when dy/dt = 0 and vertical when dx/dt = 0. This may seem faster, but sometimes can cause problems as we will soon see. 😯

Horizontal points of tangency: (0, -2) and (2, 2)

Vertical point of tangency: (-1/4, -11/8)

Let's try another with trig!

Find all points (if any) of horizontal and vertical tangency to the curve with the equations:

$x = 4\cos^2\theta$ and $y = 2\sin\theta$

$$\frac{dy}{dt} = 2\cos\theta \quad \text{and} \quad \frac{dx}{dt} = -8\cos\theta\sin\theta$$

$$\frac{dy}{dx} = \frac{2\cos\theta}{-8\cos\theta\sin\theta} = -\frac{1}{4\sin\theta}$$

$[0, 2\pi]$

So the tangent line is never horizontal.

The tangent line is vertical whenever $\sin\theta = 0$ which is at $= 0, \pi, 2\pi$

The resulting points would be: (4, 0), (4, 0) and (4, 0)

So our answer is that vertical tangency occurs at the point (4, 0).

Brain Break! It's 'Bad Joke Wednesday'!!

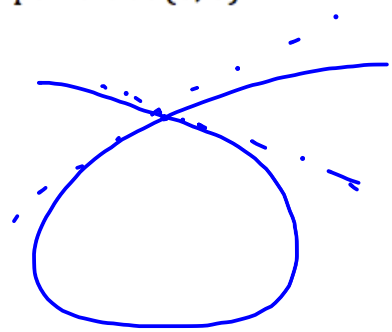
One more!!

Find the equations of the lines tangent at the point where the curve crosses itself if:

$x = t^3 - 6t$ and $y = t^2$

Hint: Don't even try to algebraically find where the curve intersects itself. Just graph it.

Based on the graph, the intersection point is at (0, 6).



$\frac{dx}{dt} = 3t^2 - 6$ and $\frac{dy}{dt} = 2t$

$\frac{dy}{dx} = \frac{2t}{3t^2 - 6}$

If x = 0 and y = 6, then:

$t^3 - 6t = 0, so\ t = 0\ and\ t = \pm\sqrt{6}$

$t^2 = 6\ so\ t = \pm\sqrt{6}$

The t's have to work for both equations so our t-values are $t = \pm\sqrt{6}$.

Slope at $t = \sqrt{6}$: $\frac{2\sqrt{6}}{18-6} = \frac{\sqrt{6}}{6}$

Slope at $t = -\sqrt{6}$: $\frac{-2\sqrt{6}}{18-6} = -\frac{\sqrt{6}}{6}$

So $y = \pm\frac{\sqrt{6}}{6}x + 6$

Alternate method for finding points of criss-cross

To do this algebraically is a bit tough:

$a^3 - 6a = b^3 - 6b \quad and \quad a^2 = b^2$

$(\pm b)^3 - 6(\pm b) = b^3 - 6b$

$-b^3 + 6b = b^3 - 6b$

$2b^3 - 12b = 0, b(b^2 - 6) = 0\ so\ b = 0\ or\ \pm\sqrt{6}$

Since b represents t, we can plug in t = 0, ±√6 to get:

(0, 0), (0, 6) or (0, 6)

Quick Reference for PVA with vectors

position vector: $\langle x(t), y(t)\rangle$

velocity vector: $\langle x'(t), y'(t)\rangle$ or $\langle dx/dt, dy/dt\rangle$

acceleration vector: $\langle x''(t), y''(t)\rangle$

speed is the magnitude of the velocity vector, so

speed = $\sqrt{(x'(t))^2 + (y'(t))^2}$

distance = $\int$|velocity| = $\int$ speed

angle = arctan(dy/dx)

slope = $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

PVA problems on the BC calculus exam usually are written as parametric equations in vector form. The math doesn't change, just the format of the answers.

Let's extend this to some AP practice!

How about 2016 #2!

What have we learned?

- Can I find the slope of a line tangent to a curve given by a pair of parametric equations?

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